Lecture 1.

Electrical engineering – basics

Introduction

Electrical engineering is, in fact, the multi-discipline, complex subject that is based on the knowledge of modern physics, mathematics, and civil engineering.

In the English-speaking countries electrical engineering includes electronics (and micro- or nano-electronics), electrical machines, power engineering, control system design, signal processing and digital signal processing (and even multidimensional signal processing, e.g. video coding), computer engineering. In Russia scope of the classic electrical engineering is folded mainly to general analysis and synthesis of lumped or distributed electrical networks.

Our course deals with linear circuit analysis and describes different techniques of steady-state (DC), sinusoidal or Fourier (AC) and time-domain analysis.

Models of electrical phenomena

We speak about electrical engineering as about technical science that studies the processes of electrical energy generation, absorption, transmission and storing and the implementations of that processes. Only the simplest models of these processes are described and relatively simple mathematical techniques are used.

It’s necessary to mention that data processing (analog or digital) and work of the control system is based on the same physical processes as power transmission, because information is obtained in a form of electrical signal and represented by current, voltage, power and energy.

Generally speaking, we form electrical network made by thousands of electrical and electronic devices connected together through cables or radio (wireless) systems and each device consists of many complicated *parts*, e.g. integrated circuits (IC), printed circuit boards (PCB), motors, generators, batteries and so on. To understand how does this system work (or why doesn’t it work), it is necessary to build a *model* of electrical energy generation, transmission and absorption in each part, device, unit and in the whole system.

Our simple models are based on different (algebraic or differential) equations on two scalar functions: current and voltage; power and energy may be introduced as non-linear combinations of these functions. Different relations between voltage and current correspond to different *elements* (resistance, capacitance, inductance). If there is a linear equation we say that the element is linear. When it is necessary to use non-linear equation, we claim this element non-linear.

Elements are connected together and form *electrical circuit*. We can combine a number of relatively simple elements to build more or less precise model of the component of the real circuit (resistor, capacitor, inductor), then form more sophisticated model of the part of the real circuit (e.g. motor), then we can form the most sophisticated model of whole system.

As the only variable time stays as an argument of the equations we speak about lumped elements electrical circuits. When time-space span of variables is taken into account we call this distributed electrical circuit.

Of cause we bound the total accuracy of our model by using two scalar functions (current and voltage), we leave all wave-propagation effects and many others. The most precise model of electromagnetic phenomena is based on quantum electrodynamics (it was developed in 1960s). This model allows to predict behavior of interacting particles: charged (electrons, protons) and non-charged particles of light (photons). However quantum models aren't useful for large-scale systems due to extremely high computational complexity.

The mid-scale (and mid-complex) model of electromagnetic phenomena is based on Maxwell's equations and it's called Maxwell's (or classic) electrodynamics. We consider this model and derive our simplified parameters (current and voltage) as integral parameters that relate to Maxwell's equations.

Maxwell's equations

Maxwell's equations are given in form:

$$\begin{cases} \left[\nabla \times \overline{H}\right] = \overline{J} + \dfrac{\partial \overline{D}}{\partial t} \\ \left[\nabla \times \overline{E}\right] = -\dfrac{\partial \overline{B}}{\partial t} \\ \left(\nabla \cdot \overline{D}\right) = \rho \\ \left(\nabla \cdot \overline{B}\right) = 0 \end{cases}$$

where:

E – vector function of the **electric field** (electric field strength), it's measured in Volt divided by meter (V/m);

D – vector function of the electric field **displacement**, it's measured in Coulomb divided by square meter (C/m^2);

H – vector function of the **magnetic field** (magnetic field strength), it's measured in Ampere divided by meter (A/m);

B – vector function of the **magnetic flux density**, it's measured in Tesla (T) or Weber divided by square meter (Wb/m^2);

J – vector function of the **conduction current density**, it's measured in Ampere divided by square meter (A/m^2);

ρ – scalar function of the **free charge density**, it's measured in Coulomb divided by cubic meter (C/m^3);

∇ – vector of differential operator "**nabla**" (Hamilton's operator, don't confuse with Hamiltonian). In Cartesian nabla has simple form:

$$\begin{cases} \left[\nabla \times \overline{H}\right] = \overline{j} + \dfrac{\partial \overline{D}}{\partial t} \\ \left[\nabla \times \overline{E}\right] = -\dfrac{\partial \overline{B}}{\partial t} \\ \left(\nabla \cdot \overline{D}\right) = \rho \\ \left(\nabla \cdot \overline{B}\right) = 0 \end{cases}$$

Normally we can use relatively simple relations between pairs of vector functions (**constitutive relations**):

$$\overline{D} = \varepsilon_0 \varepsilon_r \overline{E} = \varepsilon_a \overline{E}$$

$$\overline{B} = \mu_0 \mu_r \overline{H} = \mu_a \overline{H}$$

Together with constitutive relations Maxwell's equations form the system of linear partial differential equations (PDE) for pair of vector functions (usually they are written for *E* and *H*) with respect to time *t* and space coordinates *x*, *y* and *z*.

The very important advantage of Maxwell's equations is that they have absolutely same form in different coordinate systems; only differential operator nabla has different form in different cases.

Integral parameters are derived from the Maxwell's equations as surface, volume or loop integrals of vector functions:

$$\int_V \rho \, dV = Q$$

- **total free charge** in the volume *V* is the volume integral of ρ (charge is measured in Coulomb, C)

$$\int_S \left(\overline{J} + \frac{\partial \overline{D}}{\partial t} \right) \cdot \overline{ds} = \int_S \overline{J}_{\text{total}} \cdot \overline{ds} = i_{\text{total}}$$

- total **current** that flows through surface *S* is the surface integral of sum of conduction current density and displacement current density (current is measured in Ampere (amps, A)

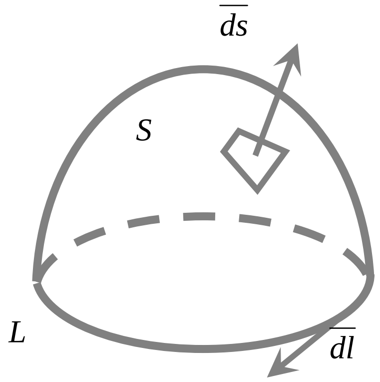
$$\int_S \overline{B} \cdot \overline{ds} = \Phi$$

- **magnetic flux** through surface *S* is the surface integral of the magnetic flux density (magnetic flux is measured in Weber, Wb)

Taking into account Stock's theorem (for the 1st and 2nd equations) and Gauss's theorem (for the 3rd and 4th equations) one can obtain integral form of the Maxwell's equations:

$$\begin{cases} \oint_L \overline{H} \cdot \overline{dl} = i_{\text{total}} \\ \oint_L \overline{E} \cdot \overline{dl} = -\dfrac{\partial \Phi}{\partial t} \\ \oint_S \overline{D} \cdot \overline{ds} = Q \\ \oint_S \overline{B} \cdot \overline{ds} = 0 \end{cases}$$

Note, that contour (loop) integral along closed loop *L* clockwise and the *dS* is pointed out of the interior that is bounded by *S*:



Voltage

In static field, when all derivatives with respect to time are equal to zero, one can substitute electric field strength *E* by the gradient of scalar function φ:

$$\overline{E} = -grad\varphi = -\nabla\varphi$$

This auxiliary function φ is called electric **potential**. Potential is measured in Volt. Potential is also an integral parameter of the electromagnetic field, because potential difference between two points is derived from the vector field *E* as a contour integral from one point to another:

$$\overline{E} = -\nabla\varphi \quad \rightarrow \quad -\int_a^b \overline{E}\,\overline{dl} = \varphi_a - \varphi_b = V_{ab}$$

This potential difference between point *a* and point *b* is also called **voltage**. We use letter *v* for the voltage but note that in Russia letter *u* is preferable.

Auxiliary potential function is valid, formally, only in static field, but electrical engineering uses simplified models of electromagnetic phenomena with time-dependent current and voltage.

Conclusion

Electrical engineering deals with simplified models of electromagnetic phenomena and uses pair of the integral parameters: current and voltage. These scalar functions are called integral parameters as they are derived as surface and contour integrals of the vector field respectively.

Lecture 2.

Electrical elements and Kirchhoff 's laws

Introduction

We found that it is possible to use reduced number of parameters of the electromagnetic field to determine a class of useful phenomena. We bound accuracy of our model by using the current and the voltage (potential) as basic variables. It is necessary to determine power, energy of the electrical signal, find models of electrical energy generation, absorption and storing by using this pair of variables. Then it is necessary to determine universal conditions for currents and voltages in the electrical circuit.

Power and energy

So, we use pair of basic variables – current $i(t)$ and voltage $v(t)$.

The instant **power** $p(t)$ is determined as a product of voltage and current:

$$p(t) = v(t)i(t)$$

Power is measured in Watt, W (Watt=Volt×Amp).

Note that direction of the current is same as the direction of the voltage. If both current and voltage are positive or negative power has positive sign. It means that electrical energy is going to the element of the circuit. If current and voltage have different signs at the same time moment, power is negative and energy goes out of the element.

Electrical **energy** is determined as integral of the instant power with respect to time:

$$W(t) = \int p(t)dt$$

Energy is measured in Joules, J (Joule=Watt×second). Total electrical energy that is absorbed (value of energy is positive) or generated (value of energy is negative) in the element may be computed as definite integral:

$$W(t) = \int_{-\infty}^{t} p(t)dt$$

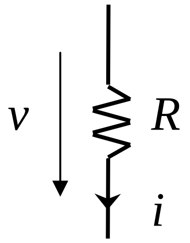
Note that the conservation of energy law leads us to the universal formula:

$$W_{total} = const$$

where W_{total} is the total energy of the system (thermal, chemical, mechanical, electrical, nuclear, etc.). If it is computed that electrical energy W during time interval Δt was absorbed in the element it means that this amount of energy was transferred into another form (e.g. dissipated as heat, thermal energy).

Electrical elements. Resistance

Consider the simplest model of the electrical energy absorption, linear **resistance**. Linear resistance *R* is the element on which voltage is proportional to current (or current is proportional to voltage):



$$v(t) = Ri(t)$$

Resistance *R* is measured in Ohm, Ω.

Sometimes it is useful to reduce reverse equation by using $g=R^{-1}$:

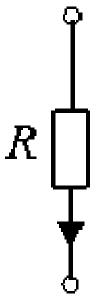
$$i(t) = gv(t)$$

Parameter *g* is called **conductance** and it is measured in Siemens, S.

Then we can rewrite **Ohm's law** or current-voltage relation for the resistance in different forms (through resistance or conductance):

$$i(t) = \frac{v(t)}{R} \quad R = \frac{v(t)}{i(t)} \quad v(t) = \frac{i(t)}{g}$$

In Russia we use another symbol for the resistance, it is shown on the following figure:



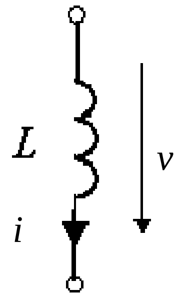
As voltage on the resistance is always proportional to the current (this is why voltage on the resistance is also called voltage drop) that flows through the resistance instant power will be always positive:

$$p(t) = v(t)i(t) = Ri^2(t) = gv^2(t)$$

It means that resistance just absorbs electrical power or one can say that the resistance is a linear model of electrical power absorption.

Electrical elements. Inductance

Inductance *L* is the element on which voltage is proportional to the first derivative of the current with respect to time:



$$v(t) = L\frac{di(t)}{dt}$$

Inductance *L* is measured in Henry, H.

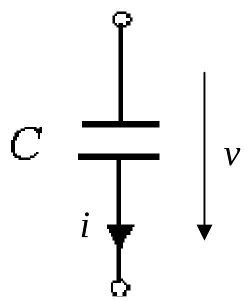
When current is rising, its first derivative is positive and the voltage on the inductance is positive respectively. Thus instant power is positive and the electrical energy is going into the inductance. When current is falling, its first derivative is negative and the voltage is negative too. In this case instant power is negative and the electrical energy is going out of the inductance.

When current doesn't vary $i(t)=I=const$ its first derivative is equal to zero, so then the voltage is equal to zero and the instant power is equal to zero. Thus the electrical energy doesn't flow into or out of the inductance. It means that inductance stores certain amount of the electrical energy (we can find that energy is stored in the magnetic field):

$$W_M(t) = \frac{Li^2(t)}{2}$$

Electrical elements. Capacitance

Capacitance *C* is the element on which current is proportional to the first derivative of the voltage with respect to time:



$$i(t) = C\frac{dv(t)}{dt}$$

Capacitance *C* is measured in Farad, F.

Capacitance works very similar to the inductance. It's just necessary to use "voltage" instead of "current" and vice versa. When voltage is rising, its first derivative is positive and the current is positive. Thus instant power is positive and the electrical energy is going into the capacitance (capacitance is being charged). When voltage is falling, its first derivative is negative and the current is negative too. In this case instant power is negative and the electrical energy is going out of the capacitance (capacitance is being discharged).

When voltage doesn't vary $v(t)=V=const$ its first derivative is equal to zero, so then the current is equal to zero and the instant power is equal to zero. Thus the electrical energy doesn't flow into or out of the capacitance. It means that capacitance stores certain amount of the electrical energy (we can find that energy is stored in the electric field):

$$W_E(t) = \frac{Cv^2(t)}{2}$$

Important notes

Energy that is stored into inductance may be evaluated as product of the current *I* and the magnetic **flux linkage** Ψ:

$$W_M = \frac{\Psi i}{2}$$

Flux linkage of the magnetic coil is equal to the magnetic flux through the turn of the coil multiplied by the number of the turns *w*:

$$\Psi = \Phi w$$

This allows us to determine inductance of the coil as the ratio between magnetic flux linkage and the current in the turns of the coil:

$$L = \frac{\Psi}{i}$$

Same things are to be said about capacitance – stored energy may be found as a product of the total charge on the plate of the capacitor and the voltage between plates:

$$W_E = \frac{Qv}{2}$$

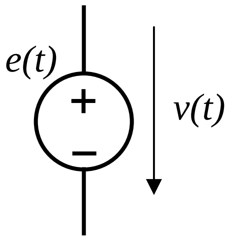
We can use it to determine the capacitance as a ratio between the charge and the voltage:

$$C = \frac{Q}{v}$$

Electrical elements. e.m.f. source

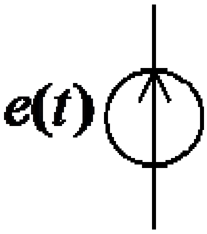
We've described models of the power absorption (resistance) and energy storing (inductance and capacitance). Models of power generation are even simpler – first model is called electromotive force source, second – current source.

Electromotive force (e.m.f.) source is the element that has voltage between terminals that is equal to given function $e(t)$:



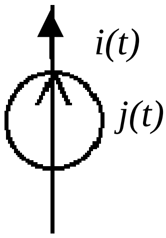
Note that this voltage (or electromotive force) doesn't depend on anything – current that flows through this element, circuit that is connected to the terminals of the source, etc. It means that power of this source is potentially unlimited. Of course, it isn't possible in the real world, so we can operate it as just an ideal model of power source, not the realistic prototype.

In Russia we use another symbol of the e.m.f. source. The most confusing thing is that this symbol is the same as US symbol of the current source.



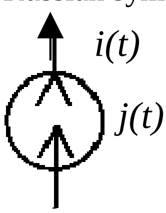
Electrical elements. Current source

Another idealistic model of power generation is the **<u>current source</u>** – element that has current flowing through it that is equal to the given function (or given value):



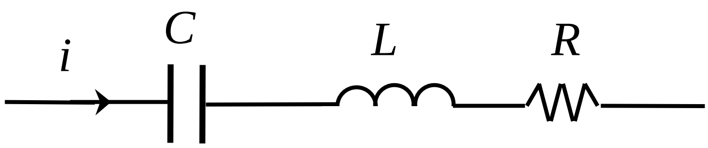
This definition means that the current that flows through this source doesn't depend on anything – voltage, external circuit that is attached to the terminals of the source, etc. This kind of source can generate infinite power, so, as well as e.m.f. source, it may be conceived as just an ideal model, not the prototype of the real device.

Russian symbol of the current source is shown on the following figure.



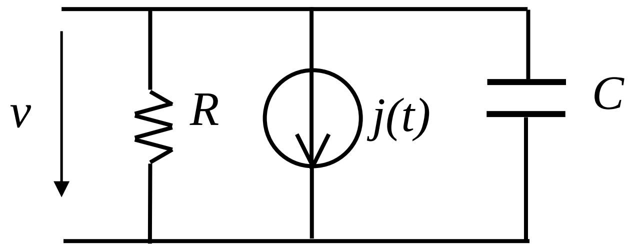
Connection of the elements. Series connection, parallel connection

We've described five electrical elements, each element has two terminals and we have relations between voltage and current for each separate element. Of course, one can connect terminals of the different elements making a circuit. If secondary terminal of the first element is connected to primary terminal of the second element and the secondary terminal of the second element is connected to the primary terminal of the third and etc. we obtain the **series connection** (see following figure).



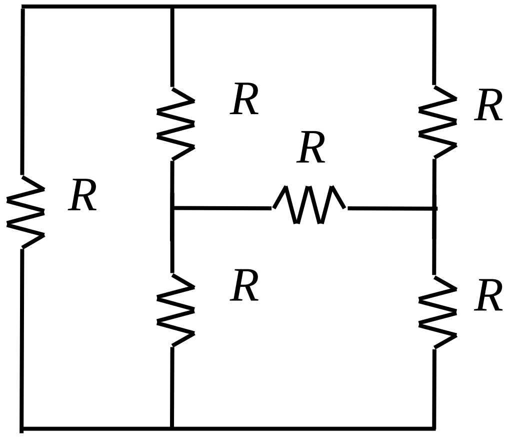
There must be the same current in each element of the series connection.

If all primary terminals of several elements are connected together and all secondary terminals of these elements are connected together (as it is shown on the figure) we obtain the **parallel connection**.



Note that there must be same voltage on all elements of the parallel connection.

Series and parallel connections are widely used (and we'll find them frequently in our problems) but it is necessary to separate series connection, parallel connection and all others (not series, nor parallel), as an example see the following figure:



Electrical circuit. Topology

Electrical **circuit** is formed by the number of electrical elements connected together. There is a number of branches, nodes and loops in the circuit.

Branch is the series connection of elements. If there is no series connection branch is formed by the only element. Number (quantity) of branches *BN* is equal to number (quantity) of different currents in the circuit.

If there is a current source in the branch current of the branch is given, known, it is equal to current source current. One can count branches with unknown currents; it determines number of equations that is necessary to solve to find all unknown currents. *CN* stays for the number of unknown currents.

Node is the point where three or more branches are connected. *NN* stays for the number (quantity) of nodes.

Loop is the continuous sequence of the branches that starts and ends at same node; loop passes each branch just once. *LN* stays for the number (quantity) of loops. Direction of the loop may be chosen arbitrary – clockwise or counter-clockwise.

Number of loops is relatively large for even simple circuits. For most purposes it is necessary to count and list not all loops but only **independent loops**. Each independent loop consists of one unique branch (this branch isn't presented in other independent loops). *IN* stays for the number (quantity) of independent loops.

Section is the imaginary line that divides circuit into two parts.

Rule: first of all count the number of unknown currents *CN*, it determines complexity of the problem**.**

Rule: when counting independent loops don't use branches with known currents (with current sources).

Important note: it is known that *CN*=*IN*+*NN*-1 (number of branches with unknown currents is equal to number of independent loops plus number of nodes minus 1).

Kirchhoff's current law (KCL)

For every node of the circuit sum of currents that flow to the node is equal to sum of the currents that flow out of the node:

$$\sum_{to\ node} i(t) = \sum_{out\ of\ node} i(t)$$

or total sum of currents in the node is equal to zero (outcoming currents are taken with "+", incoming currents are taken with "– "):

$$\sum_{node} i(t) = 0$$

Another formulation of KCL – for every section of the circuit sum of currents that flow into the section is equal to sum of the currents that flow out of the section:

$$\sum_{into\ section} i(t) = \sum_{out\ of\ section} i(t)$$

Note that KCL is valid for instant currents at each time moment.

Kirchhoff's voltage law (KVL)

For every loop of the circuit total sum of the voltages along the loop is equal to zero (if voltage has the same direction as the loop it is taken with "+", if voltage has the opposite direction – it is taken with "– "):

$$\sum_{loop} v(t) = 0$$

Note that KVL is valid for instant voltages at each time moment.

Conclusion

KCL, KVL and voltage-current relations for the elements allow us to form the system of equations (in fact, linear ordinary differential equations) and solving it find all unknown currents in the circuit. It is the base of the electrical engineering and all other things are just methods or techniques of an effective formulation and solution of such equations.

Lecture 3.

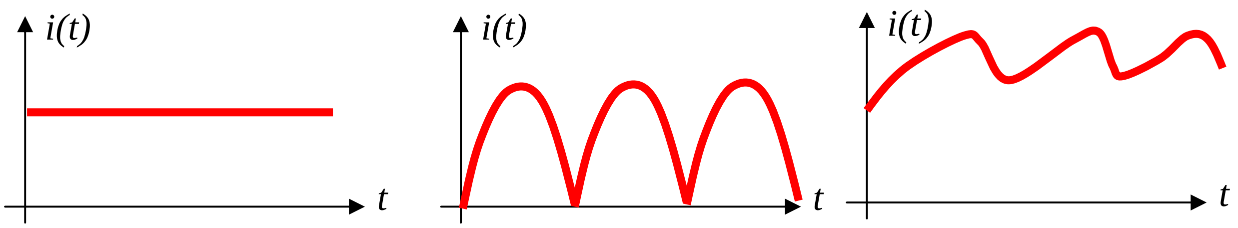
DC analysis, matrix formulation

Introduction

We've discussed Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL) as well as topological description of the circuit (branches, loops, independent loops, nodes, sections) and current-voltage (constitutive) relations for electrical elements. This lecture contains several techniques of an effective KCL and KVL equations formulation by using graphs and matrices. All methods are developed just for the DC analysis.

DC (steady-state) analysis

The abbreviation DC stays for "direct current". The strait meaning is that the current has always the same direction. Examples are shown on the following figure.



In other hand DC is widely used to show that there is time-independent, constant current that doesn't change its value. In this case "DC analysis" means steady-state or static analysis. We'll use this meaning of the DC term.

Of course, term DC should be applied to the voltage as well as to the current:

$v(t)=V=const$

$i(t)=I=const$

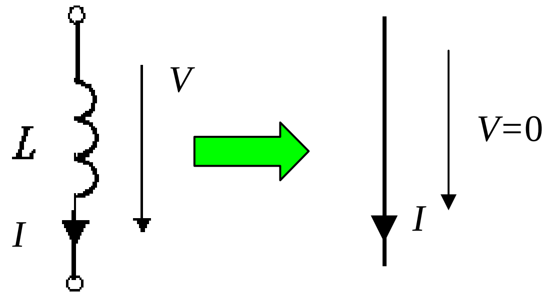
Note that we use *capital* Latin letters for constant values. This rule will be unfolded and applied to all values – e.m.f. and power.

Let's consider the behavior of different elements when all values are constants. Sources stay without changes – e.m.f. stays constant (DC e.m.f. source is frequently called battery), current of the current source remains constant.

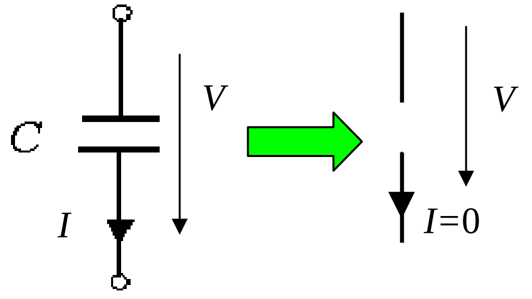
Voltage on resistance is proportional to current according to Ohm's law, so we can just rewrite it with capital letters:

$V = RI \qquad I = gV$

When constant current flows through the inductance it doesn't induce any voltage on it. It may be interpreted as that inductance has zero resistance to the constant current, or, in other words, we can substitute inductance by the **short circuit** (or short). We often draw a wire (ideal connection between terminals) instead of inductance on DC. It excludes inductances from the DC analysis but it's necessary to remember that inductance stores certain amount of energy.

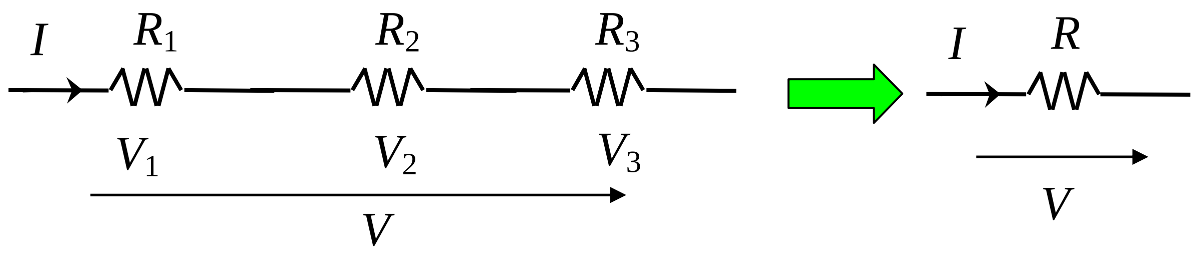


Very similar situation is for the capacitance. As voltage on it doesn't vary current is equal to zero or, in other words, DC doesn't pass through the capacitance. It may be interpreted as infinite resistance (or zero conductance) of the capacitance on DC. Thus, capacitance is often substituted by the **open circuit** (or open) – break in a wire that doesn't allow current to flow. It also excludes capacitance from the DC analysis. However it's necessary to remember that capacitance stores certain portion of energy.



KCL and KVL may be rewritten with constant currents and voltages. They lead us to the system of linear algebraic equations for currents and voltages. Then we can use Ohm's law to form the only system for currents (or for voltages).

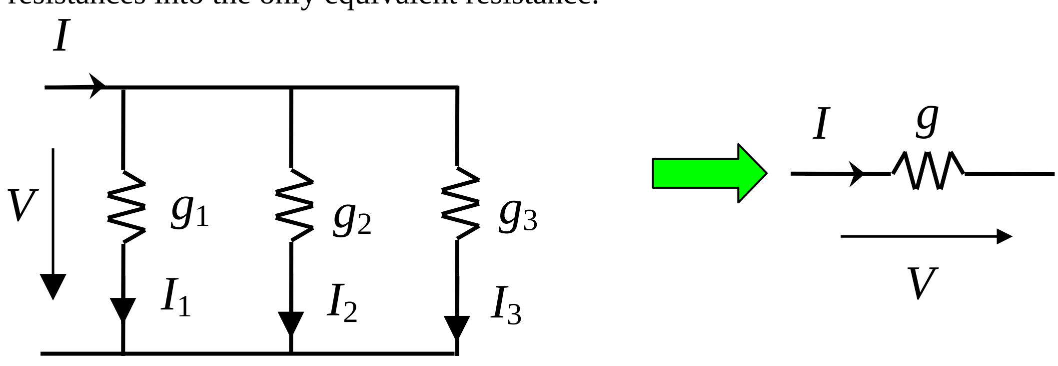
It is very useful to remember several simple formulas for DC based on KCL, KVL and Ohm's law. One is for the series connection of resistances (see the following figure).



As we have same current in each element we can sum voltage drops on resistances and use current I as a common factor to find total voltage. In brackets we have sum of the resistances. We can substitute whole sum by the only equivalent resistance that is equal to sum of all resistances:

$$V = V_1 + V_2 + V_3 = IR_1 + IR_2 + IR_3 = I(R_1 + R_2 + R_3) = IR$$

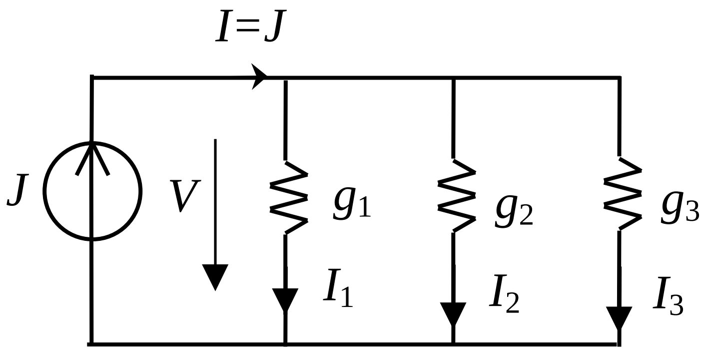
If there is parallel connection of several resistances we can apply KCL to find the total current as sum of currents in each resistance. As there is same voltage on these resistances we can find currents by using Ohm's law (it is useful to use conductances). Then total current is equal to sum of the conductances multiplied by the common voltage. It allows us to combine all parallel resistances into the only equivalent resistance.



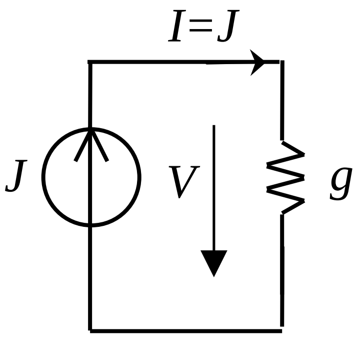
$$I = I_1 + I_2 + I_3 = Vg_1 + Vg_2 + Vg_3 = V(g_1 + g_2 + g_3) = Vg$$

So, it's easy to remember simple rule: we sum resistances for series connection and sum conductances for parallel connection.

It is important (for many practical cases) to remember the simple formula that allows to determine current in the certain branch when total current is given (we use current source J on figure to show that the total current is given).



First of all we can combine all elements together – just summing their conductances:



$$g = g_1 + g_2 + g_3$$

Then we find common voltage:

$$V = \frac{J}{g}$$

Now, we can just use Ohm's law for selected branch:

$$I_1 = Vg_1 = \frac{Jg_1}{g} = J\frac{g_1}{g_1 + g_2 + g_3}$$

$$I_2 = Vg_2 = \frac{Jg_2}{g} = J\frac{g_2}{g_1 + g_2 + g_3}$$

$$I_3 = Vg_3 = \frac{Jg_3}{g} = J\frac{g_3}{g_1 + g_2 + g_3}$$

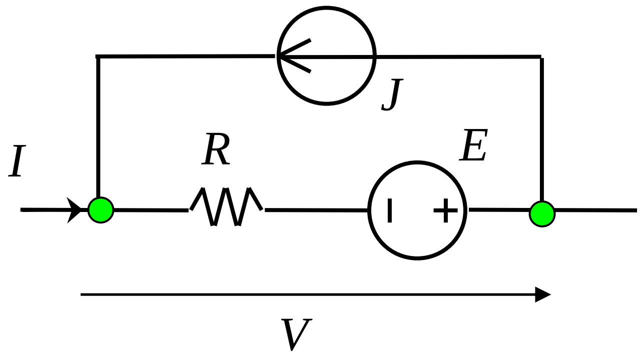
Final formula has total current as a common factor, nominator is equal to the conductance of the selected branch, and denominator is equal to sum of all parallel conductances.

Circuit graph. Matrix formulation of Kirchhoff's laws.

It isn't very difficult to use KCL and KVL for simple circuit. But there is still a problem of automated equation formation for large-scale problems. Indeed, we have to find, for example, independent loops in a circuit with hundreds of nodes and branches. There is no other way except to use matrix formulation.

Before we can use matrices it is necessary to prepare our problem – to build the **circuit graph**. Circuit graph shows interconnection of branches in original scheme. Graph consists of nodes (vertices) and branches. Vertices of the graph correspond to nodes of the original circuit. Branches of the graph correspond to **general branches** of original circuit.

General branch is formed by using the template that is shown on the figure.



General branch consists of series connection of the resistance, e.m.f. source and parallel connection of current source. Some of these elements may be equal to zero.

We can use KVL and KCL to obtain equation which connects general branch current *I* and general branch voltage *V*:

$$I = g(V + E) - J$$

or

$$V = R(I + J) - E$$

To set all the values for all general branches at once we use simple matrix formulation:

R – resistance matrix consists of the values of the resistances of general branches. R_{ii} = Resistance in the i^{th} general branch, $R_{ij} = 0$ when $i \neq j$;

G – conductance matrix, $\boldsymbol{G}=\boldsymbol{R}^{-1}$;

E – vector (column) of the e.m.f. sources in general branches, E_i = e.m.f. in the i^{th} general branch;

J – vector (column) of the current sources in general branches, J_i = current source in the i^{th} general branch.

Note that ***R*** and ***G*** are diagonal matrices.

Different dimensions of the matrices allow us to rewrite general branch equation directly in matrix form:

$$\mathbf{I} = \mathbf{G}(\mathbf{V} + \mathbf{E}) - \mathbf{J}$$

or

$$\mathbf{V} = \mathbf{R}(\mathbf{I} + \mathbf{J}) - \mathbf{E}$$

Connectivity of the general branches is given as the **nodal matrix** (or incidence matrix, or connectivity matrix):

$\boldsymbol{A}=\{a_{ij}\}$ with number of rows equal to number of nodes minus 1 and number of columns equal to number of general branches.

Nodal matrix is formed very simply:

$a_{ij} = 1$ if j^{th} general branch is connected with i^{th} node and branch current flows ***out*** of the node;

$a_{ij} = -1$ if j^{th} general branch is connected with i^{th} node and branch current flows ***into*** the node;

$a_{ij} = 0$ if j^{th} general branch isn't connected with i^{th} node.

Then, Kirchhoff's current law in a matrix form has simple notation:

$\mathbf{AI} = 0$

Of course, this equation mayn't be solved as number of rows isn't equal to number of columns.

Another formulation of KCL deals with the sections of the circuit. To obtain sufficient number of sections it is necessary to build **tree** of the graph. Tree is a part of the graph which connects all nodes and doesn't consist of any loops. There are many computational techniques in discrete math that form tree of the graph. The simplest one is the exclusion of the branches one by one with the check of connectivity. Number of branches in every tree of the graph is the same and it is equal to number of nodes minus one.

All branches that aren't included in the tree are called connectivity branches. Each connectivity branch corresponds to the section (this sequence of sections is called main sections). We can form matrix of the main sections and rewrite KCL in the other form:

$\mathbf{QI} = 0$

Mesh matrix shows independent loops of the graph:

$\boldsymbol{B}=\{b_{ij}\}$ with number of rows equal to number of independent loops and number of columns equal to number of general branches.

Mesh matrix is formed very similarly to nodal matrix:

$b_{ij} = 1$ if j^{th} general branch is attached to i^{th} loop and branch current flows in the ***same*** direction as a direction of the loop;

$b_{ij} = -1$ if j^{th} general branch is attached to i^{th} loop and branch current flows in the ***opposite*** direction;

$b_{ij} = 0$ if j^{th} general branch isn't attached to i^{th} loop.

Then, Kirchhoff's voltage law in a matrix form has simple notation:

$\mathbf{BV} = 0$

After all we can combine matrix formulations of KCL and KVL and matrix equation of the general branch:

$$\begin{cases}\mathbf{AI}=0\\ \mathbf{BV}=0\end{cases}\rightarrow\begin{cases}\mathbf{A}(\mathbf{G}(\mathbf{V}+\mathbf{E})-\mathbf{J})=0\\ \mathbf{BV}=0\end{cases}\rightarrow\begin{cases}\mathbf{AGV}=\mathbf{AJ}-\mathbf{AGE}\\ \mathbf{BV}=0\end{cases}$$

- there is system of the linear algebraic equations which may be solved.

Or, another order of the operations gives us:

$$\begin{cases}\mathbf{AI}=0\\ \mathbf{BV}=0\end{cases}\rightarrow\begin{cases}\mathbf{AI}=0\\ \mathbf{B}(\mathbf{R}(\mathbf{I}+\mathbf{J})-\mathbf{E})=0\end{cases}\rightarrow\begin{cases}\mathbf{AI}=0\\ \mathbf{BRI}=\mathbf{BE}-\mathbf{BRJ}\end{cases}$$

Conclusion

DC analysis deals with only constant currents and voltages. Inductances and capacitances are excluded in the steady-state. We can use Ohm's law, KCL and KVL to solve any DC problem. Circuit graph and the matrix formulation of the equations give us the formal technique of the DC analysis.

Lecture 4.

Nodal analysis, mesh analysis. Superposition principle. Reciprocity.

Introduction

We've found that matrix formulation of KCL and KVL leads us to the formal solution of the linear system of algebraic equations. Hence, computational complexity of this method is rather high. Indeed, it is necessary to solve 2*n* equations (n is the number of general branches), because we have to evaluate branch currents and branch voltages simultaneously. Computational complexity of the method is $4n^2$. Note, that if we know branch currents (*n* currents) we can easily find all voltages through general branch equation, so it isn't necessary to double number of equations. Another important thing is the linearity of our equations. This property allows to introduce very general principles that are useful for electrical circuit analysis.

Nodal analysis

To obtain fewer equations we can use auxiliary variables (it is very common mathematical technique). For example, let's consider potentials of the nodes (**nodal potentials**) φ_i. Voltage of the general branch between nodes *i* and *j* is equal to the potential difference:

$$V=\varphi_i-\varphi_j$$

We can use nodal matrix to rewrite this statement in a matrix form:

$$\mathbf{V}=\mathbf{A}^T\varphi$$

Therefore, we can substitute branch voltages in general branch matrix equation by the nodal potentials:

$$\mathbf{I}=\mathbf{G}(\mathbf{V}+\mathbf{E})-\mathbf{J}\quad\rightarrow\quad\mathbf{I}=\mathbf{G}(\mathbf{A}^T\varphi+\mathbf{E})-\mathbf{J}=\mathbf{GA}^T\varphi+\mathbf{GE}-\mathbf{J}$$

Consequently, we apply matrix form of KCL and obtain matrix equation:

$$\mathbf{AI} = 0 \quad \rightarrow \quad \mathbf{AGA}^T\varphi + \mathbf{AGE} - \mathbf{AJ} = 0 \quad \rightarrow \quad \varphi = \left(\mathbf{AGA}^T\right)^{-1}\left(\mathbf{AJ} - \mathbf{AGE}\right)$$

This is the linear system of the algebraic equations of the order of number of nodes minus 1. This technique is called **<u>nodal analysis</u>**.

This method includes following steps:

1) form the circuit graph;
2) fill matrices **R,G,E,J**;
3) find nodal matrix **A**;
4) compute $\varphi = \left(\mathbf{AGA}^T\right)^{-1}\left(\mathbf{AJ} - \mathbf{AGE}\right)$;
5) compute $\mathbf{V} = \mathbf{A}^T\varphi$;
6) compute $\mathbf{I} = \mathbf{G}(\mathbf{V} + \mathbf{E}) - \mathbf{J}$;

The computational complexity of the nodal analysis is proportional to square of number of nodes.

There was formal way of the nodal analysis introduction. This method is very convenient tool in electrical engineering and electronics, therefore it is very useful to remember simple rule how to form nodal equations without matrices.

Rule: left-hand expression: potential of the node is multiplied by the sum of conductances of the branches attached to this node. Then we subtract potentials of the neighbor nodes that are multiplied by the sum of conductances between our node and neighbor nodes. Right-hand expression: sum of the e.m.f. sources that are attached to the node (if positive terminal is attached to the node – sign is "+", if negative terminal is attached – sign is "–"), multiplied by their conductances; sum of the current sources (if positive terminal of the current source is attached to the node – sign is "+", otherwise – "–").

Mesh analysis

As we used auxiliary variables (nodal potentials) in the nodal analysis it is possible to introduce auxiliary variables for branch currents – **<u>mesh currents</u>**:

$$\mathbf{I} = \mathbf{B}^T\mathbf{I}_M$$

The general branch matrix equation may be rewritten in a following form:

$$\mathbf{V} = \mathbf{R}(\mathbf{I} + \mathbf{J}) - \mathbf{E} \quad \rightarrow \quad \mathbf{V} = \mathbf{R}\left(\mathbf{B}^T\mathbf{I}_M + \mathbf{J}\right) - \mathbf{E} = \mathbf{RB}^T\mathbf{I}_M + \mathbf{RJ} - \mathbf{E}$$

Then, KVL matrix equation is rewritten as:

$$\mathbf{BV} = 0 \quad \rightarrow \quad \mathbf{BRB}^T\mathbf{I}_M + \mathbf{BRJ} - \mathbf{BE} = 0 \quad \rightarrow \quad \mathbf{I}_M = \left(\mathbf{BRB}^T\right)^{-1}\left(\mathbf{BE} - \mathbf{BRJ}\right)$$

Therefore, we obtain the linear system of the algebraic equations of the order of number of independent loops. This technique is called **<u>mesh analysis</u>**.

This method includes following steps:

1) form the circuit graph;
2) fill matrices **R,G,E,J**
3) find mesh matrix **B**
4) compute $\mathbf{I}_M = \left(\mathbf{BRB}^T\right)^{-1}\left(\mathbf{BE} - \mathbf{BRJ}\right)$
5) compute $\mathbf{I} = \mathbf{B}^T\mathbf{I}_M$
6) compute $\mathbf{V} = \mathbf{R}\left(\mathbf{I} + \mathbf{J}\right) - \mathbf{E}$

The computational complexity of the nodal analysis is proportional to square of number of independent loops.

Historical note: both techniques (nodal analysis and mesh analysis) were introduced by James C. Maxwell in his “Treatise on electricity and magnetism”.

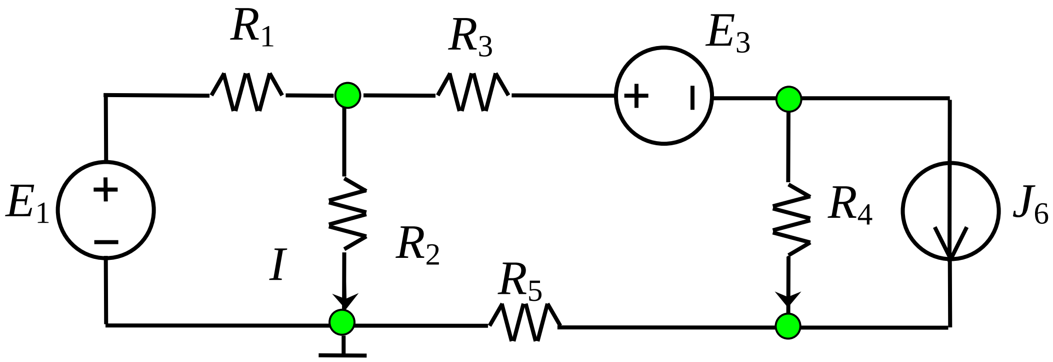
Another remark: note, that both nodal analysis and mesh analysis deal with only one matrix of the circuit graph (nodal and mesh respectively). It means that the least information required about the circuit may be collected in the one matrix (nodal or mesh). But it doesn’t mean that we can operate only KCL or KVL – auxiliary variables were introduced to satisfy KVL (nodal potentials) or KCL (mesh currents) “by default”.

Superposition principle

As we found earlier, we’ve obtained linear systems of algebraic equations. KCL and KVL are both linear combinations of voltages and currents. And as all the elements of the circuit are linear whole circuit may be considered as linear system. **Superposition principle** is the very general principle over all linear systems (electrical, mechanical, optical, thermal, etc.) – reaction of the linear system caused by the several stimuli is the *sum of the partial reactions* of the system caused by each stimulus individually.

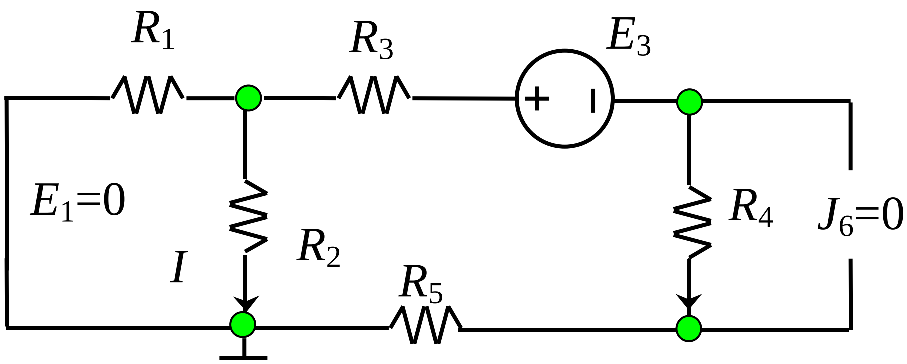
In electrical engineering superposition principle has the following form: every current (or voltage) in the circuit may be represented as the sum of **partial currents** (or **partial voltages**) caused by the each source separately.

For example, to find partial current I caused by the source E_3 (see the scheme on the following figure)



it is necessary to:

- set all other sources to zero (turn them off);



- compute partial current I in the simplified circuit.

$$I = \frac{E_3}{R_2 \parallel R_1 + R_3 + R_4 + R_5} \frac{g_2}{g_1 + g_2}$$

Note that setting e.m.f. source to zero is the same thing as to substitute it by the short circuit (indeed, voltage on the turned off e.m.f. source is equal to zero as the voltage drop on the short circuit); setting current source to zero is the same as to substitute it by the open circuit (current is equal to zero as the current source is turned off and, in other hand, current doesn't pass through the open circuit).

Of course, setting all sources with the only exception to zero significantly simplify the circuit. Thus, it is relatively simple problem to find the partial current.

We can use formal expression for the superposition principle:

$$I_i = \sum_j g_{ij} E_j + \sum_j k_{ij} J_j$$

In the linear circuit partial current is proportional to the e.m.f. source which caused it with the coefficient g_{ij} – **partial mutual conductance** (this coefficient is measured in Siemens as ordinary conductance element). Here i is a number of the branch with the current and j is a number of branch with the source. When source is in the same branch as the current $i=j$ and g_{ii} is called **partial self conductance**.

Although we call this coefficient conductance it isn't the model of power absorption so it might be negative or positive depending on the direction of the current.

If the partial current is connected with current source coefficient in the proportion k_{ij} is called **partial current transmission coefficient**. It is the dimensionless value and it may be in the range from -1 to +1.

Formal expression of the superposition principle for the voltage is given as following:

$$V_i = \sum_j k^V{}_{ij} E_j + \sum_j R_{ij} J_j$$

Partial mutual resistance R_{ij} and **partial self resistance** R_{ii} are introduced similarly to partial conductances – as ratio of the partial voltage and the current source.

Of course, this coefficient is measured in Ohm.

Ratio $k^v{}_{ij}$ between the voltage in i^{th} branch and e.m.f. source in j^{th} branch is called **partial voltage transmission coefficient**. It is the dimensionless coefficient and it may have value in the range from -1 to +1.

Matrix formulation of the superposition principle is given in form:

$$\begin{cases} \mathbf{I} = \mathbf{K_I J} + \mathbf{G_I E} \\ \mathbf{V} = \mathbf{K_V E} + \mathbf{R_V J} \end{cases}$$

and combines matrices of all possible coefficient for all branches.

Reciprocity principle

Formal introduction of the **reciprocity principle** is based on the matrix formulation of the superposition principle:

$$\mathbf{G_I} = \mathbf{G_I}^T, \quad \mathbf{K_I} = \mathbf{K_I}^T, \quad \mathbf{R_V} = \mathbf{R_V}^T, \quad \mathbf{K_V} = \mathbf{K_V}^T$$

This expression means that all partial coefficient matrices are *symmetrical*.

In other words $g_{ij}= g_{ji}$, $k_{ij}= k_{ji}$, $k^v{}_{ij}= k^v{}_{ji}$, $R_{ij}= R_{ji}$.

If a circuit satisfies this principle it is called **reciprocal circuit**.

As reciprocity is based on the superposition one can conclude that:

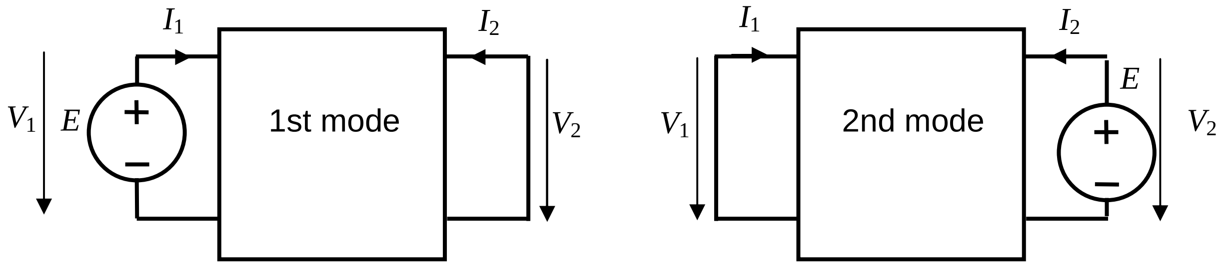
- every reciprocal circuit is linear;
- linear circuit isn't reciprocal definitely.

There is another definition of the reciprocity principle – if there are two selected branches in the reciprocal circuit the following expression take place:

$$I_i^{(1)} V_i^{(2)} + I_j^{(1)} V_j^{(2)} = I_i^{(2)} V_i^{(1)} + I_j^{(2)} V_j^{(1)}$$

where $I_i^{(1)}, V_i^{(1)}, I_i^{(2)}, V_i^{(2)}, I_j^{(1)}, V_j^{(1)}, I_j^{(2)}, V_j^{(2)}$ – voltages and currents in *i*-th and *j*-th branches in two different *modes*. Difference between these two modes is that different sources are turned on and off.

Simple example shows properties of the reciprocal circuit:



First mode: there is e.m.f. source E in the first branch and the short circuit is in the second. Second mode: there is e.m.f. source E in the second branch and the short circuit is in the first. In accordance with the reciprocity principle:

$$I_1^{(1)}V_1^{(2)} + I_2^{(1)}V_2^{(2)} = I_1^{(2)}V_1^{(1)} + I_2^{(2)}V_2^{(1)} \quad \to \quad I_1^{(1)}0 + I_2^{(1)}E = I_1^{(2)}E + I_2^{(2)}0 \quad \to \quad I_2^{(1)} = I_1^{(2)}$$

or, source in the first branch induces the same short current in the second branch as same source induces in the first branch being placed into the second branch.

It is important principle that is used, for example, in measurements.

Conclusion

Although KCL, KVL and Ohm's law give us the straight way to solve any problem we need, it is important to reduce complexity of the computations. The pair of auxiliary variables leads us to the pair of the efficient techniques – nodal analysis and mesh analysis. Which method is preferable? We should use the method that gives the least number of equations. It is necessary to compare the number of nodes and number of independent loops. If number of nodes minus 1 is less than the number of independent loops, nodal analysis is preferable. Otherwise, mesh analysis is better.

Linear formulation of basic laws allows us to use powerful tool of the analysis – superposition principle. It has numerous applications in different methods, it reduces the complexity of a problem, it allows to compute and analyze responses separately. And another powerful analysis tool is based on it – the reciprocity principle.

Lecture 5.

Real source, its equivalent schemes. Linearity. Power balance.

Introduction

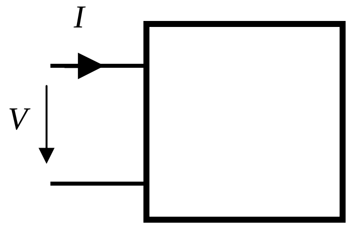
As it was said at the beginning of the course, the key question of the electrical engineering is the generation of the electric power. Ideal models (e.m.f. source and current source) are used for circuit analysis but the real process of electric power generation is still in question. So the model of the real power source is the first point of this lecture. Then we'll discuss properties of these models, their applications, e.g. Norton's theorem and Thevenin's theorem.

Volt-amps diagram

The starting point of the lecture is the **volt-amp diagram**, important tool of the DC analysis. Generally speaking, it's not only the diagram; it's also the general relation between voltage and current on the element with two terminals. Sometimes it is called **volt-amps characteristic** to underline the importance of the general functional representation. In practice, there is no real need to plot the diagram for most of linear cases but it plays an important role in the non-linear circuit analysis.

Volt-amps diagram is the dependence of between the voltage and the current on the element with two terminals:

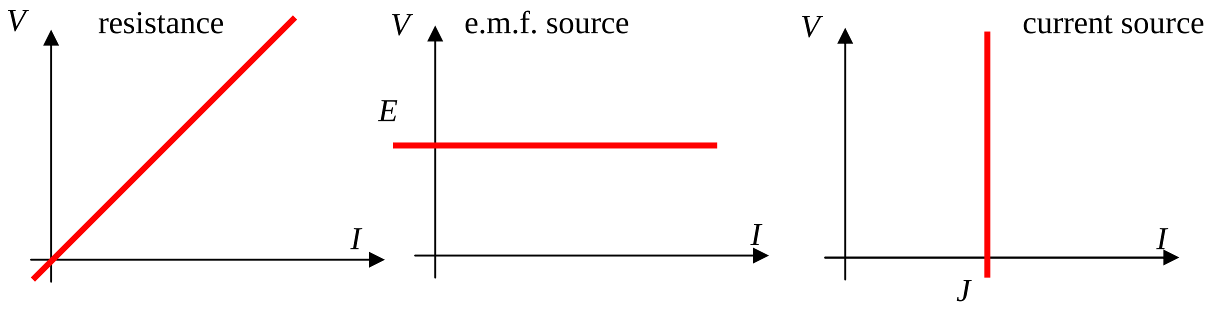
$V(I), \quad I(V)$



In linear circuits the only available form of the functional dependence is linear:

$V(I) = aI + b = -R_{diff} I + V_0, \quad I(V) = cV + d = -g_{diff} V + I_0$

Volt-amps diagrams for ideal elements of the electrical circuit are shown on the following figure:



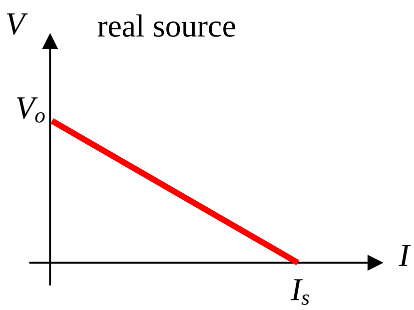
$V = RI \quad I = gV$

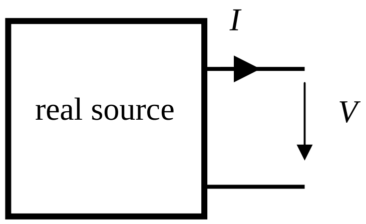
Coefficient R_{diff} of the linear function is called **differential resistance**; it is equal to the first derivative of the voltage with respect to current. It is constant for all linear elements. Differential resistance of the ideal element resistance coincides with resistance R. Differential resistance of the e.m.f. source is equal to zero. Differential resistance of the current source approaches infinity.

First derivative of the current with respect to voltage is called **differential conductance**. In linear cases g_{diff} is always in reverse proportion to R_{diff}. So, differential conductance of the ideal resistance coincides with conductance g. Differential conductance of the current source is equal to zero. Differential conductance of the e.m.f. source approaches infinity.

Real power source

Real power source has limited capacity. Power that may be generated by this source can’t exceed certain limit. The only possible linear function that gives us proper volt-amps diagram of the real source is as following:





$$V(I) = -R_{diff} I + V_o, \quad I(V) = -g_{diff} V + I_s$$

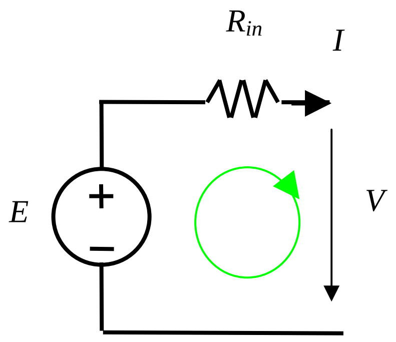
Note that the voltage V on the terminals of this source doesn’t exceed V_o and the current I that is flowing out of the source is limited by value I_s.

Indices o and s stand for *open* and *short* respectively. V_o is the voltage on the open terminals when no current flow through it. I_s is the short current when the output voltage of the source is equal to zero.

It will be shown later that the power of such source is limited by the value of $0{,}5V_o I_s$ that is, in fact, the square under the volt-amps diagram plot.

The next problem is to make a connection between ideal electrical elements and the linear mathematical model of the real power source or, in other words, find an equivalent scheme which satisfies equation of the real power source.

It is evident that the pair of coefficients of the linear equation corresponds to the pair of ideal elements. Let’s try series connection of the e.m.f. source E and the resistance R_{in} (index *in* stands for *internal*):



This circuit consists of the only branch with current I and the only loop (there are no nodes in this circuit). KVL may be used to obtain an equation:

$$-E + R_{in} I + V = 0$$

Moving parts of this expression one can find the final equation:

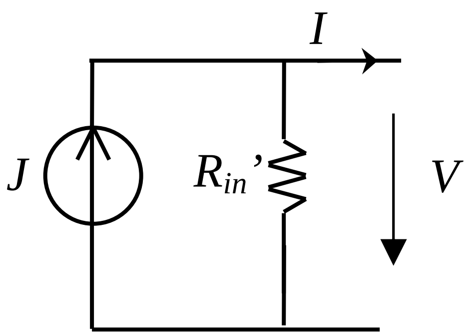
$$V = E - R_{in} I$$

which corresponds to linear equation of the real power source.

This scheme is called **series equivalent scheme** of the real power source.

Parameter E is equal to V_o and $R_{in} = R_{diff}$. Short current $I_s = E / R_{in}$.

Parallel equivalent scheme may be introduced similarly. Consider parallel connection of the current source J and the resistance R_{in}':



This circuit consists of three branches (one of them (with output current I) goes through the external circuit), two nodes, three independent loops. KVL indicates the same fact as a definition of the parallel connection: the output voltage V is applied to all three branches simultaneously. KCL leads to the following equation:

$$J = \frac{V}{R_{in}'} + I$$

It is conveniently to use conductance g_{in}' in this expression:

$$J = Vg_{in}' + I \quad \rightarrow \quad I(V) = -g_{in}'V + J$$

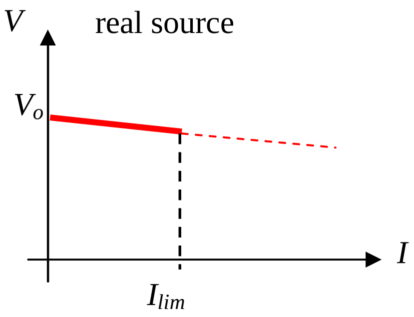
Thus, we obtain equation which corresponds to real power source equation. Parameter J is equal to I_s and $g_{in}' = g_{diff}$. Open voltage $V_o = J / g_{in}'$.

There is an **equivalence conditions** that connect parameters of different schemes of the real power source. Indeed, there is only *one* source, *one* diagram and *two* possible schemes. As both schemes correspond to the same diagram equivalence conditions are:

$$R_{in}' = R_{in}$$

$$V_o = E = I_S R_{in}$$

Sometimes engineers speak about "real voltage (or e.m.f.) source" or "real current source". It is just a choice between two equivalent circuits – series scheme consists of e.m.f. source and may be referred as "real e.m.f. source", parallel scheme includes current source and is frequently called "real current source". Which scheme is preferable? Note, that we can always use any of them under the equivalence conditions. Nevertheless, real source which is built to keep at almost constant voltage (e.g. battery) works properly in the relatively narrow range of the output current (see the following figure; current is limited by the value of $I_{lim} << I_s$). In this case $V/I = R_{load} \approx V_o/I_{lim}$ and $R_{in} = V_o/I_s$ and, finally, $R_{load} >> R_{in}$.



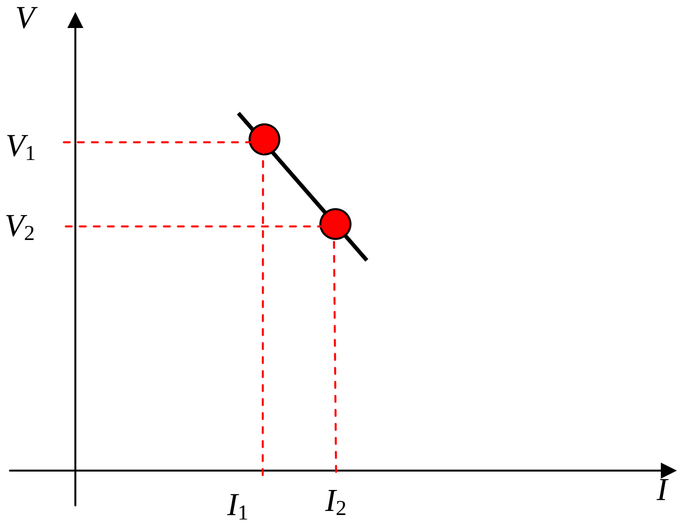
Thus, series scheme is preferable when internal resistance of the source significantly less than the load resistance (load resistance models power absorption in the external circuit).

Otherwise, real source may be built to keep the certain value of the current (e.g. power sources for the motors). Such source works properly in the narrow range of the output voltages and the output voltage is limited by $V_{lim} << V_o$. So, $R_{load} \approx V_{lim}/I_s$ and $R_{load} << R_{in}$. One can say that the parallel scheme is preferable when internal resistance of the source is significantly greater than load resistance.

When load resistance has approximately same value as internal resistance, equivalent scheme may be chosen behind the reason of computational complexity reducing.

There is an important note here. Experimental observation of the volt-amps diagram of the certain source should be started from the choice of the proper model. It is necessary to select a reasonable range of the load resistance value, because, for example, connecting source with small internal resistance to small external (load) resistance it is possible to obtain very large current (at almost short current) which exceeds I_{lim} and may harm source. When the "real current source" (source with relatively high value of the internal resistance) is left with disconnected terminals (so that is the open circuit is used as a load) voltage between terminals will rise dramatically and it may damage the source.

It is interesting that thermal power plant behaves as a real source but in the very narrow range of the output power. Output voltage may vary between V_1 and V_2 and the output current – between I_1 and I_2 . Open circuit on the output isn't allowed because it is necessary to transport the energy from the turbine and generator somehow and somewhere or to stop the power plant immediately. Short circuit is a kind of disaster. Parameters of the equivalent schemes may be computed by using of the pair of the points of the given volt-amps diagram:



$$R_{in} = \frac{V_1 - V_2}{I_2 - I_1}, \quad V_o = V_1 + I_1 R_{in}$$

Input resistance

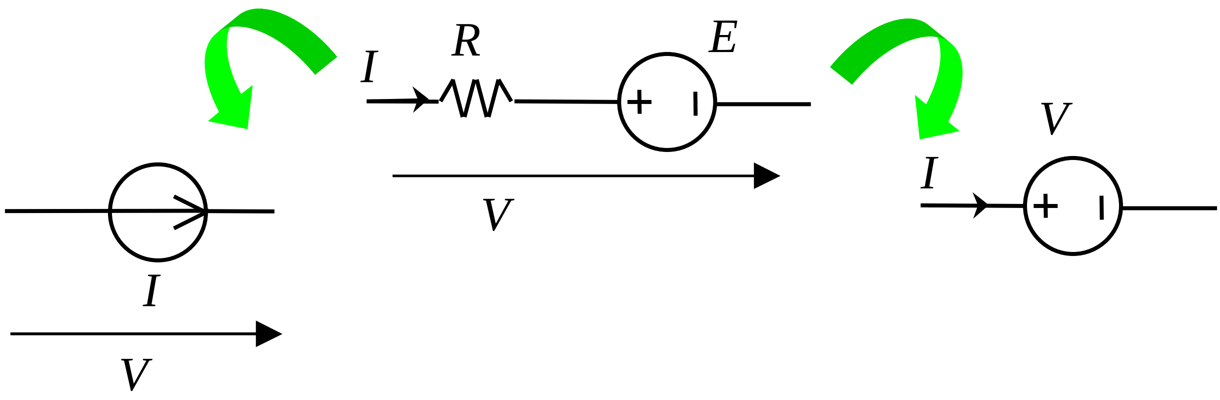
Note that we used resistance R_{load} as a model of the power absorption in the external circuit. Internal structure of this circuit doesn't matter as we are interested in the power source mode only. Resistance R_{load} may be computed as a ratio of the voltage on the terminals of the circuit to the current that flows through the terminals. This pair of terminals may be referred as the *input* of the circuit or they may be called *input terminals*. Important thing is that we assumed that external circuit just absorb and doesn't generate electric power.

Circuit which doesn't generate electric power (or doesn't consist of the sources) is called **passive circuit**. Otherwise, circuit that consists of the sources is called **active circuit**.

The common definition is that the **input resistance** is a ratio of the input voltage to input current of the *passive* circuit. Input voltage is the voltage on the input terminals, input current is the current that flows through input terminals. So, resistance R_{load} is in fact input resistance of the external circuit. Note that we can find the input resistance of the real power source – but first, it is necessary to make this circuit passive that is to set e.m.f. source (or current source) to zero. Then, input resistance may be found as R_{in} (note that index *in* may be interpreted as *input* as well as *internal*) for both equivalent schemes.

Compensation principle

Compensation principle may be applied to reduce complexity of the electrical circuit when certain information is available about the circuit mode. If the current in the branch is known (given, measured, already computed) whole branch may be substituted by the current source with the known current (see the left figure). If the voltage on the branch is known (given, measured, already computed) whole branch may be substituted by the e.m.f. source with the known voltage (see the right figure).



Linearity principle

As a real power source may be represented by the linear volt-amps equation, every voltage (or current) in the linear circuit may be represented as a linear function of any other voltage (or current). This is the **linearity principle** which is based on the superposition or, better to say, is one and the same as superposition principle.

Following expressions show the linearity principle:

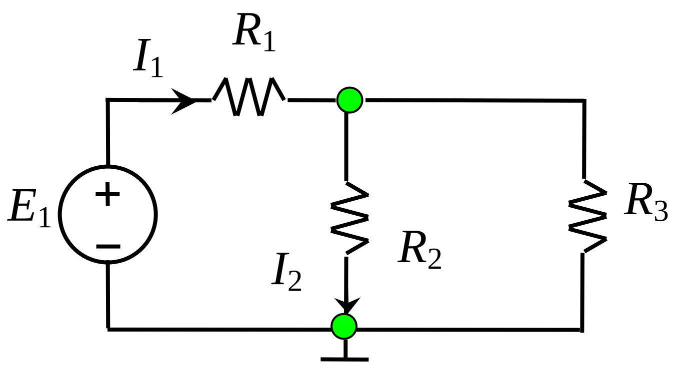
$$I_i = aI_j + b$$

$$I_i = cV_j + d$$

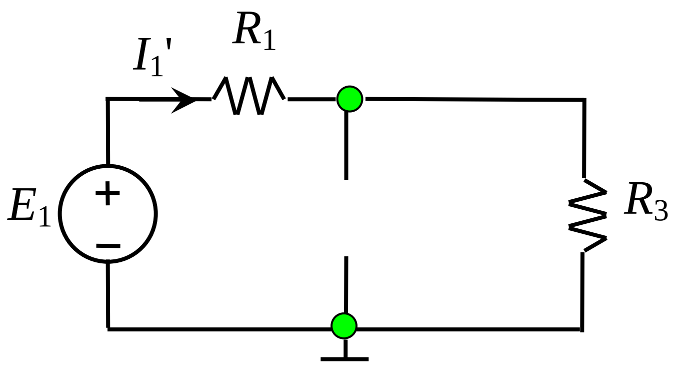
$$V_i = eV_j + f$$

$$V_i = hI_j + k$$

The following example shows how to find coefficients of the linear proportion between two currents in a circuit.

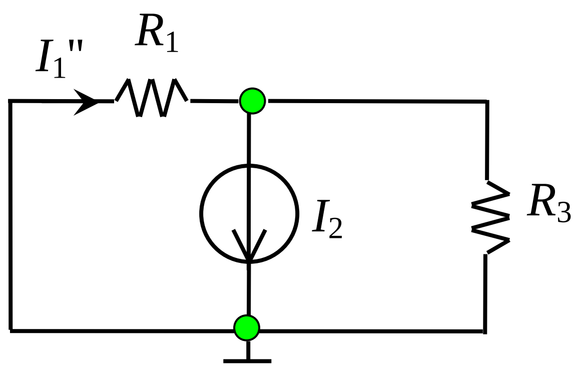


At first, let's consider circuit with the open circuit instead of the branch with the current I_2. Current I_1' determines coefficient b in the linear function. As the only loop stays in this scheme, KVL gives:



$$I_1' = \frac{E_1}{R_1 + R_3} = b$$

Then, according to the superposition principle, all sources are to be set to zero (turned off) and, according to the compensation principle, branch with the current is to be substituted by the current source $J = I_2$. Ratio I_1''/I_2 determines coefficient a of the linear function. Current I_1'' may be found by using KVL: KVL shows that there is the same voltage V on the branches; we can sum parallel conductances and find the voltage V:



$$V = \frac{I_2}{g_1 + g_3}$$

$$I_1'' = \frac{I_2 g_1}{g_1 + g_3}$$

Then, current I_1'' is equal to Vg_1.

Final linear formula is as following:

$$I_1 = \frac{I_2 g_1}{g_1 + g_3} + \frac{E_1}{R_1 + R_3}$$

Thevenin's theorem and Norton's theorem

Consider situation when linearity principle is applied to voltage and current in the same branch:

$$V_i = hI_i + k$$

This expression has the same form as the equation of the real power source. This allows to formulate theorem: if there is a selected branch with current I and voltage V the *rest part of the circuit* may be represented by the *real power source* with two equivalent schemes and parameters of this source are open voltage, short current and input resistance of the rest part of the circuit.

To find the open voltage it is necessary to substitute selected branch by the open circuit and compute the voltage on it.

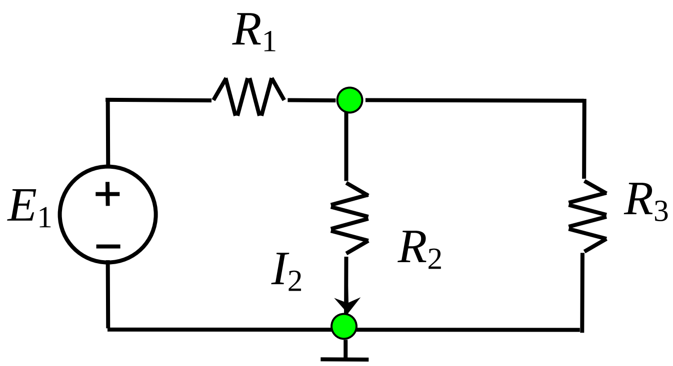
To find the short current it is necessary to substitute selected branch by the short circuit and compute current in it.

To find the input resistance it is necessary to make the rest part of the circuit *passive* and somehow compute input resistance (e.g. combining series and parallel connections of the resistances).

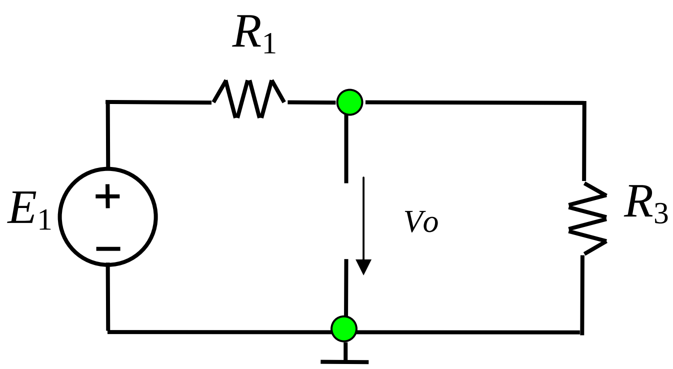
Note that three listed parameters don't form the independent system – one of them may be found through others by using the equivalence condition.

Representation of the rest part of the circuit by the series equivalent scheme is called **Thevenin's theorem**, representation by the parallel equivalent scheme is called **Norton's theorem**. This is why series scheme is often called **Thevenin's scheme** and parallel scheme is often called **Norton's scheme**.

Following example shows the implementation of the theorem:

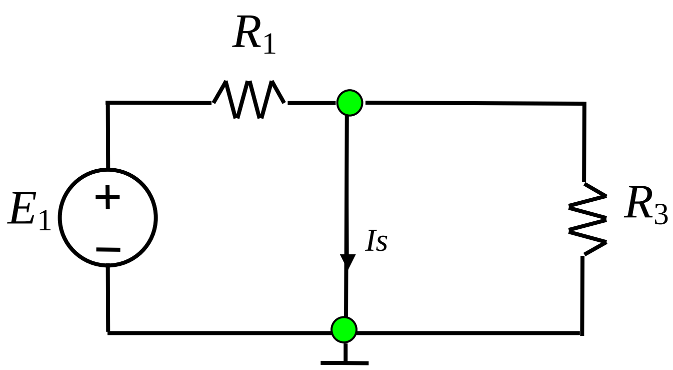


1) Let’s find the open voltage. Second branch is substituted by the open circuit; there is one branch and one loop left in the circuit. KVL gives us open voltage as:



$$V_o = \frac{E_1 R_3}{R_1 + R_3}$$

2) Second branch is substituted by the short circuit. Note that current flows through the e.m.f. source E_1, resistance R_1, short circuit and doesn’t flow through R_3. KVL leads us to:



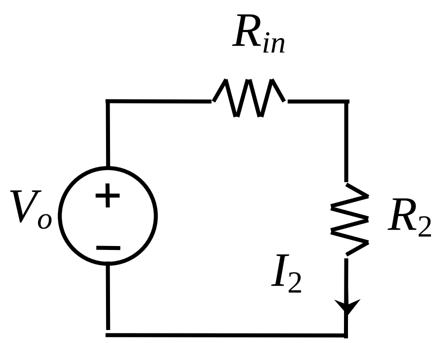
$$I_s = \frac{E_1}{R_1}$$

3) Let's find input resistance independently – set e.m.f. source E_1 to zero, then find the input resistance: $R_{in}=R_1||R_3$.

Note that:

$$\frac{V_o}{R_{in}} = \frac{E_1R_3}{R_1+R_3}\frac{R_1+R_3}{R_1R_3} = \frac{E_1}{R_1} = I_s$$

After all we can substitute rest part of the circuit by the equivalent scheme, e.g. series equivalent scheme:



There is only one loop in this circuit so unknown current may be found as:

$$I_2 = \frac{V_o}{R_{in}+R_2}$$

Power balance

<u>Power balance</u> simply corresponds to the conservation of energy law. General form of this principle is that the total *generated* power is equal to the total *absorbed* power in a circuit:

$$P_{generated} = P_{absorbed}$$

It is understandable that total absorbed power should be computed as a sum of power absorbed in every resistance. Ohm's law claims that voltage and current have the same direction on the resistance, so resistance always absorbs power:

$$P = RI^2 = gV^2$$

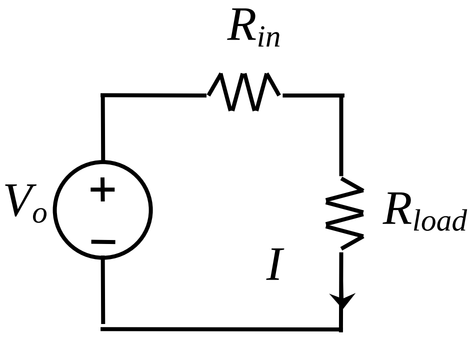
Generation of the power is the process when current and voltage on the terminals of the element have *different* direction. Consider e.m.f. source – if current flows out from positive terminal and returns back to the negative terminal power is generated. Otherwise, power is absorbed in the e.m.f. source, or in other words e.m.f. source works against other sources reducing total generated power (e.g. recharge of the battery). Similarly, if the voltage on the current source drops from positive terminal to negative power is generated. If voltage has another direction (voltage *rises* from negative terminal to positive) power is absorbed.

Thus, power balance may be written in the following form:

$$\sum VJ + \sum EI = \sum RI^2$$

Maximal power transmission

Sometimes it is necessary to use the maximal possible amount of the electric power from the given real source (e.g. battery, DC generator, secondary power supply adapter, etc.). It is assumed that we can adjust the load (R_{load}) to obtain maximal energy in specified period of time.



Mathematically it is formulated as following:

$$R_{in} = ? \quad \rightarrow \quad Max(P_{load})$$

Of course, we can compute load power as $I^2 R_{load}$. But as the R_{load} is to be found and the current I depends on it, this formulation isn't successful. It is better to use power balance to compute load power as a difference of generated power and wasted (absorbed) power in R_{in}:

$$P_{load}(I) = V_0 I - R_{in} I^2$$

This formula consists of only one unknown variable I and allows to use well known technique: extremum (maximum or minimum) is at the point where first derivative is equal to zero:

$$Max(P_{load}(I)) \rightarrow \frac{dP_{load}(I)}{dI} = 0$$

$$\frac{dP_{load}(I)}{dI} = V_0 - 2R_{in} I$$

Thus, current value that corresponds to maximal load power is equal to:

$$I = \frac{V_o}{2R_{in}}$$

KVL gives us formula for the current:

$$I = \frac{V_o}{R_{load} + R_{in}}$$

Comparison of our results allows to conclude that maximal power is gained when load resistance is equal to internal resistance of the source:

$$R_{load} = R_{in}$$

Note that **efficiency** of this process is equal to 50% because the same power is absorbed on the internal resistance (this power is *dissipated* inside the source, sometimes it overheats the source and may cause serious damage) and on the load resistance. Of course, this efficiency is absolutely insufficient for the industrial power transmission lines, power plants and etc. If the efficiency is the critically important parameter of a system (e.g. due to economical reasons) it is necessary to keep the great difference between load resistance and input resistance.

Conclusion

It was last lecture on DC analysis. Further materials will be more or less complicated but very many things will have very similar formulation; we'll use same techniques, methods, principles and theorems with just a little mathematical difference. And, of course, DC analysis itself is an important part of the transient analysis.

There are several topics that we have left out of the scope of our course – triangle-star transform, advanced techniques of nodal analysis, etc. These topics are included into advanced courses of an electrical engineering.

Lecture 6.

AC analysis. Phasor currents. Complex impedance.

Introduction

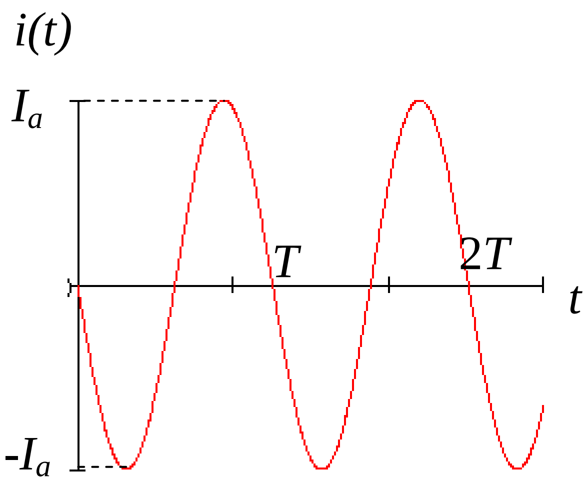
We are starting a new part of our course. We'll find this part very similar to the DC analysis with one significant difference – DC analysis works with the *real* numbers, but AC analysis deals with the *complex* numbers. There will be a brief introduction into the complex numbers in this lecture. Another difficulty is the large number of new terms: amplitude, magnitude, phase, impedance, admittance, reactance, susceptance and etc.

AC analysis

AC stands for the *alternating current*. There are two points of view on the term AC: current that alternates (changing its direction) in time and sinusoidal current. AC analysis always relates to the second meaning – AC analysis deals with sinusoidal waveforms of the currents and the voltages.

Why do we use sinusoidal waveform not rectangular or triangle? There are two reasons. First reason is that the sinusoidal waveform is the natural waveform of a voltage (or current) which is easy to generate. Sinusoidal waveform of the voltage is induced by the constant magnetic field in the rotating coil (angular speed of the coil is assumed to be constant). Second reason is in the mathematical technique called Fourier analysis. At almost every possible waveform of the signal may be represented by the sequence of sin waves.

Let's consider the current that has sinusoidal waveform:



$$i(t) = I_a \sin(\omega t + \varphi)$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

$$f = \frac{1}{T}$$

here T – **period** of the current (period is measured in second, s);

f – **frequency** of the current (frequency is measured in Hertz; Hz);

ω – **angular frequency** (angular frequency is measured in $second^{-1}$, s^{-1}, not in Hz);

φ – **starting phase** (usually referred as **phase**), (phase is measured in radians, rad, or degrees, °);

I_a **– amplitude** of the current (amplitude is measured in Ampere, A).

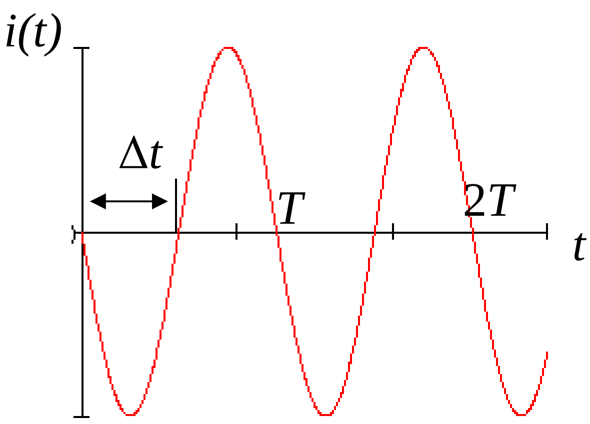
Note: don't confuse radians and degrees in the argument of the sin function.

Sinusoidal voltage is described similarly:

$$v(t) = V_a \sin(\omega t + \psi)$$

To differentiate parameters of the voltage and the current it is assumed that the phase of the voltage is ψ.

The phase of the sinusoidal current (or voltage) may be determined by the time delay Δt before the starting point of the sin function ($sin(0)=0$ and $sin'(0)>0$):



$$\varphi = 360° \frac{\Delta t}{T} = 360° \Delta t f$$

Electrical elements behavior

Let's consider voltage-current relation for different elements and determine dependencies between parameters of the sinusoidal functions.

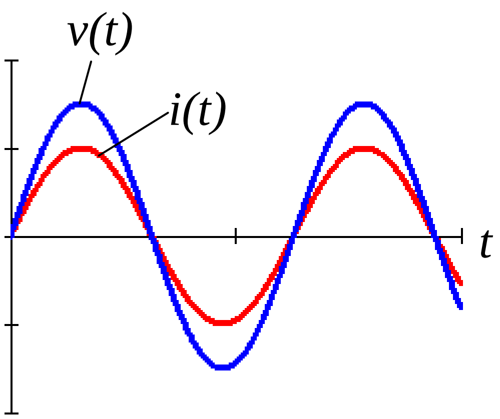
Resistance (conductance) – Ohm's law is valid for instant voltage and current, so we can just multiply amplitude of the current by *R* to obtain amplitude of the voltage. Phase of the voltage is the absolutely same as the phase of the current.

$$v(t) = Ri(t) = RI_a \sin(\omega t + \varphi)$$

$$V_a = I_a R$$

$$\psi = \varphi$$

Following figure shows this simple fact.



Inductance – there is differential relation between voltage and the current:

$$v(t) = L\frac{di(t)}{dt} = LI_a \frac{d}{dt}\sin(\omega t + \varphi)$$

Note that *sin'*(*ax*)=*a cos*(*ax*):

$$v(t) = LI_a \omega \cos(\omega t + \varphi)$$

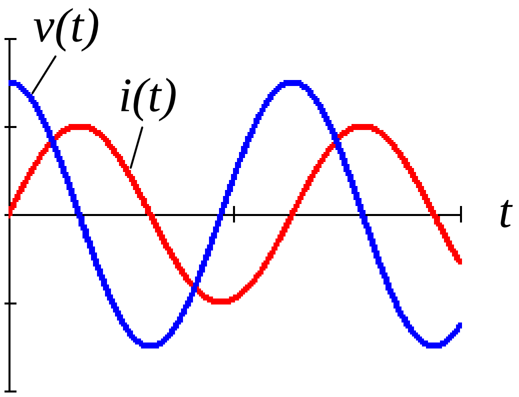
As AC analysis is based on the uniform sinusoidal representation, it is necessary to substitute *cos*(*x*) by *sin*(*x*+90°):

Thus, amplitude of the voltage is proportional to amplitude of the current, inductance and the angular frequency:

$$v(t) = I_a L\omega \sin(\omega t + \varphi + 90°)$$
$$V_a = I_a L\omega$$
$$\psi = \varphi + 90°$$

Phase of the voltage is 90° ahead than the phase of the current. Engineers say that the voltage is **leading** and the current is **lagging**.

The following figure shows current and voltage on the inductance.



Capacitance – the behavior is very similar to the inductance, but it is necessary to exchange voltage and current. Indeed, current is proportional to the first derivative of the voltage:

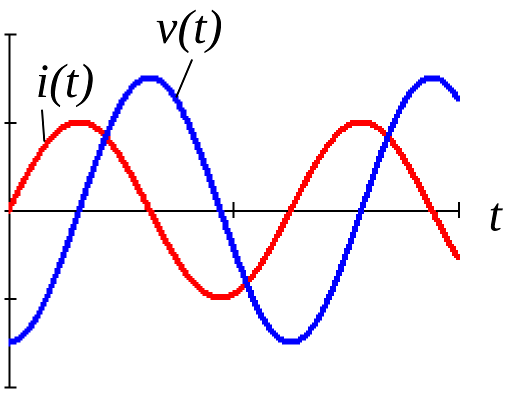
$$i(t) = C\frac{dv(t)}{dt} = CV_a \frac{d}{dt}\sin(\omega t + \psi)$$

Amplitude of the current is proportional to amplitude of the voltage, capacitance and the angular frequency:

$$i(t) = V_a C\omega \sin(\omega t + \psi + 90°)$$
$$I_a = V_a C\omega$$
$$\psi = \varphi - 90°$$

Current is leading, voltage is lagging and the phase difference is 90°:

The following figure shows current and voltage on the capacitance.



One can conclude that amplitude of the voltage is proportional to the amplitude of the current in all three cases. There are different coefficients, different dependencies on the angular frequencies and different phases. It is important that all voltages and currents keep the sinusoidal waveform and period (frequency).

Extract from the complex numbers theory

This chapter describes the most useful formulas and terms of the complex numbers theory.

Complex number has two parts: **real part** and **imaginary part**:

$$z = x + jy$$

x – real part, *y* – imaginary part, *j* – **imaginary unit**. This representation of the complex number is called algebraic form.

In mathematics there is commonly used letter *i* for the imaginary unit, but in electrical engineering *i* is used for current.

Of course, you remember that $j^2 = -1$.

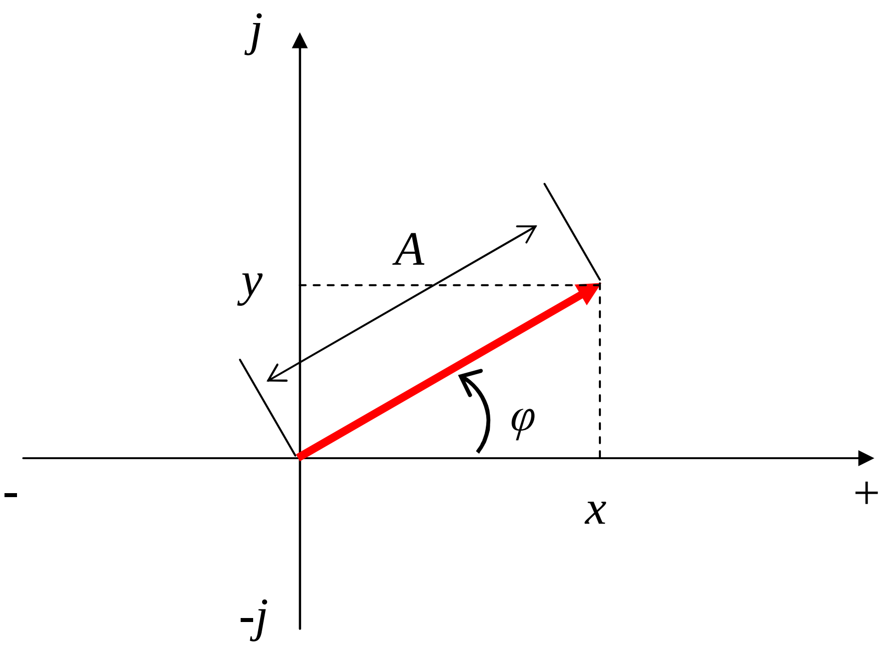
Euler's formula gives us another representation of the complex number – the polar form:

$$Ae^{j\varphi} = A\cos\varphi + jA\sin\varphi$$

A – **magnitude** of the complex number;

φ – **argument** of the complex number.

Complex number may be represented as a point on the **complex plane** or vector - **phasor**:



Abscise axis is the real axis, ordinate axis – imaginary. So that real part is the abscissa of the point and imaginary part is the ordinate. Length of the vector that points from the point (0,0) to the point which correspond to the complex number is equal to the magnitude of the complex number. Angle between abscise axis and this vector is equal to the argument of the complex number (note that the positive direction of the angle is *counterclockwise*).

It is useful to apply short abbreviations and special signs to point out different forms and parameters of a complex number:

Real part: $x = \mathrm{Re}(z) = A\cos\varphi$

Imaginary part: $y = \mathrm{Im}(z) = A\sin\varphi$

Magnitude (absolute value): $A = |z| = \sqrt{x^2 + y^2}$

Argument: $\varphi = \arg(z) = \arctan\left(\frac{y}{x}\right)$

Note that we use ratio of *y* to *x* as an argument of arctangent function and this ratio has the positive sign when *x* and *y* both have same sign (doesn't matter positive or negative) and the ratio has negative sign when *x* and *y* have different signs (doesn't matter which one is positive). This results as an argument displacement:

$$\arctan\left(\frac{y}{-x}\right) = \arctan\left(\frac{-y}{x}\right)$$

$$\arg(-x + jy) = \arg(x - jy) + 180°)$$

The following table helps to compute argument:

$x > 0$	$y > 0$	φ∈[0, 90°]	$\varphi = \arctan\left(\frac{y}{x}\right)$
$x > 0$	$y < 0$	φ∈[0, -90°]	$\varphi = \arctan\left(\frac{y}{x}\right)$
$x < 0$	$y > 0$	φ∈[90°, 180°]	$\varphi = \arctan\left(\frac{y}{x}\right) + 180°$
$x < 0$	$y < 0$	φ∈[180°, 270°]	$\varphi = \arctan\left(\frac{y}{x}\right) + 180°$

This problem is avoided in the special software (e.g. Mathcad ®, Matlab ®, Scilab ®, etc.) which includes specialized functions of an argument computation.

There are simple rules of summation, multiplication and division of the complex numbers:

$$z_1 = x_1 + jy_1 = A_1 e^{j\varphi_1} \quad z_2 = x_2 + jy_2 = A_2 e^{j\varphi_2}$$

$$z_1 \pm z_2 = (x_1 \pm x_2) + j(y_1 \pm y_2)$$

$$z_1 z_2 = A_1 A_2 e^{j(\varphi_1 + \varphi_2)}$$

$$\frac{z_1}{z_2} = \frac{A_1}{A_2} e^{j(\varphi_1 - \varphi_2)}$$

Note that it is strongly recommended to use algebraic form for summation (subtraction) and polar form for multiplication (division). Polar form is also preferable for power function and roots extraction:

$$z^n = A^n e^{jn\varphi}$$

$$\sqrt{z} = \sqrt{A} e^{j\frac{\varphi}{2}}$$

There is a specific operation that is defined only for the complex numbers (not for the real numbers) – **conjugation**. Complex number z^* is the conjugated complex number z, when:

$$z = x + jy = Ae^{j\varphi} \quad z^* = x - jy = Ae^{-j\varphi}$$

Complex representation. Phasor current, phasor voltage

It is evident that different complex numbers have different pairs of parameters (magnitude and argument or real part and imaginary part). In other hand, if two sinusoidal currents have the same frequency, difference between them is in the pair of parameters – amplitude and phase. This simple fact leads us to the idea that it is possible to use complex number (each complex number has two unique parameters) as a representation of the sinusoidal currents of the same frequency (each current has two unique parameters). One can mention that this representation looses waveform and frequency, but it is assumed that the waveform of all currents and voltages is the same – sinusoidal and, similarly, that all currents and voltages have the same frequency. Thus waveform and frequency aren't unique property that can be used to discriminate currents (or voltages). Formal way to the complex representation is more sophisticated:

$$i(t) = I_a \sin(\omega t + \varphi) = \mathrm{Im}(I_a \cos(\omega t + \varphi) + jI_a \sin(\omega t + \varphi)) = \mathrm{Im}\left(I_a e^{j(\omega t + \varphi)}\right) = \mathrm{Im}\left(\dot{I}_a(\omega t)\right)$$

One can use real part of the complex number to make a link between complex number and the current. There may be theory of *cosinusoidal* currents and it is very close to our AC analysis. Choice of the imaginary part is just a kind of a treaty inside the society of electrical engineers. This uncertainty simply shows that complex numbers just represent currents (and voltages) but don't have any physical meaning.

Complex current consists of the *exp*($j\omega t$) part that is exactly the same for all complex currents of the same frequency. Thus one can leave this common part (reduce it in every equation) and use the **complex amplitude** or **phasor current** only:

$$\dot{I}_a = I_a e^{j\varphi} = \frac{\left(\dot{I}_a(\omega t)\right)}{e^{j\omega t}}$$

Here argument of complex amplitude corresponds to the phase of the current and magnitude – to the amplitude of the current.

Note that the formal return to the sinusoidal function of time includes attachment of the complex exponent *exp*($j\omega t$).

Absolutely same representation is available for the voltage (**phasor voltage**):

$$\dot{V}_a = V_a e^{j\psi}$$

Note that we use dot above the capital Latin symbol to tell the phasors (with dot) and amplitude (without dot) of the current.

KCL in a complex form has the same formulation as for DC analysis – sum of the complex currents (phasor currents) that flows into the node is equal to sum of the complex currents (phasor currents) that flows out of the node:

$$\sum_{\text{int}\,o\ node} \dot{I}_a = \sum_{out\ of\ node} \dot{I}_a$$

KVL is also available in the ordinary form – sum of the complex voltages (phasor voltages) along the loop is equal to zero:

$$\sum_{along\ loop} \dot{V}_a = 0$$

Impedance and admittance

First of all, let's check the Ohm's law formulation with complex amplitudes. As it was shown earlier, the voltage has the same phase as the current and amplitude of the voltage is proportional to resistance and amplitude of the current:

$$\dot{V}_a = RI_a e^{j\varphi} = R\dot{I}_a$$

It means that Ohm's law has absolutely same form as for the DC analysis.

Then, it was found that the phase of the voltage on the inductance is 90° ahead in comparison to the current's phase; amplitude of the voltage is proportional to inductance, amplitude of the current and the angular frequency:

$$\dot{V}_a = j\omega L I_a e^{j\varphi} = j\omega L \dot{I}_a$$

One can combine these coefficients (angular frequency, phase difference and inductance) into one complex coefficient Z_L:

$$Z_L = j\omega L$$

$$\dot{V}_a = Z_L \dot{I}_a$$

Consequently, we obtain complex formula that is similar to the Ohm's law. The only difference is that we use complex coefficient Z_L instead of resistance R. This similarity allows us to use Z_L as *complex resistance* which corresponds to inductance at frequency ω. In Russia we use exactly "*complex resistance*" but in the English-speaking countries and commonly all over the world there is a special term – **impedance**.

Of course, impedance is measured in Ohms.

Ohm's law may be written in other form – trough conductance g. Similarly, one can write:

$$Y_L = \frac{1}{j\omega L}$$

$$\dot{I}_a = Y_L \dot{V}_a$$

where Y_L – "complex conductance" or **admittance** of the inductance.

Same things are to be said about capacitance. The phase of the current on the capacitance is 90° ahead in comparison to the voltage phase; amplitude of the current is proportional to capacitance, amplitude of the voltage and the angular frequency:

$$\dot{I}_a = j\omega C V_a e^{j\psi} = j\omega C \dot{V}_a$$

Combining angular frequency, phase difference and inductance into one complex coefficient we obtain admittance of the capacitance Y_C:

$$Y_C = j\omega C$$

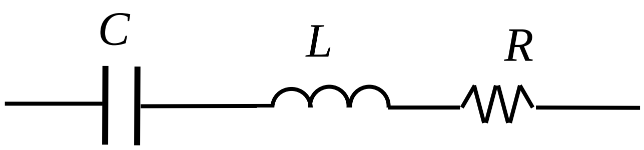
$$\dot{I}_a = Y_C \dot{V}_a$$

Impedance of the capacitor may be found easily:

$$Z_C = \frac{1}{j\omega C}$$

$$\dot{V}_a = Z_C \dot{I}_a$$

If there series connection of several elements is presented we sum complex impedances of these elements:



$$Z = Z_C + Z_L + R = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

Note that we obtained complex number with non-zero real and imaginary parts. So, the real part of the complex impedance is called resistance (as the electrical element) and the imaginary part of the complex impedance is called **reactance**.

Similarly, if there is the parallel connection of several elements it is convenient to sum complex admittances:

$$Y = Y_C + Y_L + g = g + j\left(\omega C - \frac{1}{\omega L}\right)$$

The real part of the admittance is called conductance and the imaginary part is called **susceptance**.

Note that there is reverse proportion between impedance and admittance, not between resistance (as part of impedance) and conductance (part of admittance) nor between reactance and susceptance.

Another note to remember – impedance (admittance) is complex number but reactance, resistance, conductance ant susceptance are the real numbers.

And, finally it is important that impedance always refers to the certain frequency of the sinusoidal electrical signal. If there are several sources with different frequencies or non-sinusoidal sources it is impossible to use impedances (admittances) of elements for all presented sources simultaneously. Natural parameters of elements are capacitance and inductance, not impedance.

Conclusion

The focus point of the lecture is the introduction of the very useful technique of the AC analysis that allows us to deal with complex algebra not with the differential equations. It is important that complex numbers represent currents and voltages or electrical element at the certain frequency. In fact, frequency is the general (global) parameter of the AC analysis.

Lecture 7. Average power. Complex power. RMS current. Power balance.

Introduction

Electrical elements were introduced as simple models of energy absorption (resistance), energy generation (current source, e.m.f. source) and energy storing (capacitance, inductance). AC analysis uses complex numbers for currents, voltages and impedances of the elements. Connection of the power transformations with the complex technique of the AC analysis is in the scope of the lecture.

Average power

As it was introduced above, instant power is the product of the instant voltage and instant current:

$$p(t) = v(t)i(t)$$

When voltages and currents have the same frequency and same (sinusoidal) waveform, we obtain instant power as sum of two functions – constant and sinusoidal with doubled frequency:

$$p(t) = V_a \sin(\omega t + \psi) I_a \sin(\omega t + \varphi) = \frac{1}{2}\left(V_a I_a \cos(\psi - \varphi) - V_a I_a \cos(2\omega t + \psi + \varphi)\right)$$

Sinusoidal part of the instant power changes polarity four times during the period T that means that one quarter of period power is “charging” the element, next quarter of the period power is “discharging”, then “charge” and “discharge” processes are going one by one. Of course, the constant part of the instant power shows the process of power generation or absorption. This constant part may be found as an **<u>average power</u>** during the period:

$$P = \frac{1}{T}\int_0^T p(t)dt = \frac{V_a I_a}{2T}\int_0^T \left(\cos(\psi - \varphi) - \cos(2\omega t + \psi + \varphi)\right)dt = \frac{V_a I_a}{2}\cos(\psi - \varphi)$$

Note that we use capital Latin *P* for the average power as for the DC power. Indeed, this value shows the amount of energy which is absorbed or generated by the element during the period so it is the rate of the total energy flow as DC power is. Averaging during period eliminates variable part of the instant power. Therefore, charge-discharge process isn't taken into account and this process has no influence on the total energy flow.

The important thing is that the average power depends on the phase difference between the voltage and the current. It doesn't matter which one is leading or lagging because $cos(x) = cos(-x)$, so $cos(\psi-\varphi) = cos(\varphi-\psi)$.

Note that $cos(90°) = 0$, so $\psi-\varphi = 90°$ means that there is no total energy flow at all. It was shown previously that the voltage is 90° leading on the inductance and 90° lagging on the capacitance. Thus, there is zero average power on such elements, or, in other words, capacitances and inductances don't store or absorb or generate power, they just become charged and discharged twice during the period. It may be claimed that inductances and capacitances don't form any part of the total energy flow.

This result is derived from the general time-domain formulas of the sinusoidal voltages and currents. Average power depends on amplitudes and phases of the voltage and current, so one can suppose that it is possible to use complex representation to obtain general but complex expression for the power.

The problem is that the complex representation gives, of course, complex result for all kinds of voltage and current products.

For example we can obtain multiplication of magnitudes and phase difference in the formula by using following substitutions:

$\dot{V}_a\left(\dot{I}_a\right)^* \quad \dot{I}_a\left(\dot{V}_a\right)^*$

It is relatively simple to understand what to do with the real part of the complex power – being associated with the average power it shows the total energy flow. Total energy is the subject of the conservation of energy law so it may be easily found that there is an average power balance in the circuit. Imaginary part of the complex power isn't connected with the total energy flow so there is no point to find physical basics of “imaginary power balance” or “imaginary energy conservation”. It is necessary to find and prove that the complex power (introduced by the certain way) is balanced in the circuit. This connection is based on the Tellegen's theorem.

Tellegen's theorem

Tellegen's theorem has very wide and general application – it is used in electrical, thermal and hydro engineering, data processing, network theory, economic modeling, and society analysis. Theorem assets that if there are two conditions given:

$\boldsymbol{AI=0}$ and $\boldsymbol{V=A^T\varphi}$

the product $\boldsymbol{V^T I}$ is equal to zero.

Proof:

$$\mathbf{V}^T\mathbf{I} = \left(\mathbf{A}^T\varphi\right)^T\mathbf{I} = \varphi^T\mathbf{A}\mathbf{I} = 0$$

Note that the first statement is the matrix form of KCL and the second condition is the introduction of the nodal potentials. Thus, every electrical circuit satisfies these conditions. And the complex representation satisfies the requirements of the theorem – it may be written in many different forms:

$$\dot{\mathbf{V}}_{\mathbf{a}}{}^T\dot{\mathbf{I}}_{\mathbf{a}} = \mathbf{0} \quad \dot{\mathbf{V}}_{\mathbf{a}}{}^T\left(\dot{\mathbf{I}}_{\mathbf{a}}\right)^* = \mathbf{0} \quad \left(\dot{\mathbf{V}}_{\mathbf{a}}{}^T\right)^*\dot{\mathbf{I}}_{\mathbf{a}} = \mathbf{0}$$

One can choose

$$\dot{\mathbf{V}}_{\mathbf{a}}{}^T\left(\dot{\mathbf{I}}_{\mathbf{a}}\right)^* = \mathbf{0}$$

or

$$\left(\dot{\mathbf{V}}_{\mathbf{a}}{}^T\right)^*\dot{\mathbf{I}}_{\mathbf{a}} = \mathbf{0}$$

Therefore, one of these equations may be considered as complex power balance for the AC analysis of the electrical circuit. Compare this result with the $\boldsymbol{V^T I} = 0$ which was obtained for the DC analysis.

Complex power balance

Let's consider the following definition of the **complex power**:

$$\tilde{S} = \frac{1}{2}\dot{V}_a\left(\dot{I}_a\right)^* = P + jQ$$

Tellegen's theorem leads us to the matrix formulation of the **complex power balance**:

$$\frac{1}{2}\dot{\mathbf{V}}_{\mathbf{a}}{}^T\left(\dot{\mathbf{I}}_{\mathbf{a}}\right)^* = \mathbf{0}$$

The sign "tilde" above capital Latin *S* means that it is complex power but in contrast to complex current (or complex voltage) it corresponds to non-sinusoidal function (as it is found above, instant power has two parts – constant and sinusoidal with doubled frequency).

The real part of the complex power coincides with average power and it is also called **active power**. As it is complained above active power corresponds to total energy flow. If active power is positive, energy is absorbed in the element; if it is negative positive, energy is generated in the element. We use Watt as a basic unit for an active power measurement.

The imaginary part of the complex power is called **reactive power**. Note that reactive power depends on $sin(\psi$-$\varphi)$, so the sign of the reactive power is determined by the sign of the phase difference – if voltage leads (e.g. voltage on inductance) it is positive, if the current leads (e.g. current in capacitance) – it is negative. Reactive power is measured in VAR (volt-amps-reactive)

– physically same dimension as Watt, it is introduced to differentiate parts of the complex power. Similarly, special unit is introduced to measure complex power – VA (volt-amps).

There is no reasonable physical explanation – what does the reactive power means. Generally speaking, reactive power shows (under certain circumstances) the efficiency of the electrical energy transportation (e.g. from power plant to consumer). The maximal possible efficiency of the power transmission may be determined by the absolute value of the complex power. In this case *cos*(ψ-φ) has very similar meaning (but not same!) to the efficiency coefficient.

There is another possible choice of the complex power:

$$\tilde{S} = \dot{I}_a \left(\dot{V}_a\right)^*$$

This choice leads to the same expressions for the active power as *cos*(ψ-φ) = *cos*(φ-ψ) and changes sign of the reactive power as *sin*(ψ-φ) = - *sin*(φ-ψ). Thus reactive power of the inductance is negative and reactive power of the capacitance is positive. This formulation is commonly used by engineers in the US, previous one – in Europe and Russia. As reactive power doesn't have its own physical meaning the problem of the choice of the sign of the reactive power doesn't imply practical or theoretical difficulties.

Power balance may be formulated in other form: sum of the complex power that is generated by the sources is equal to the sum of the complex power on each passive element.

$$\sum_{generated} \tilde{S} = \sum_{absorbed} \tilde{S}$$

or

$$\frac{1}{2}\sum \dot{E}_a \left(\dot{I}_a\right)^* + \frac{1}{2}\sum \dot{V}_a \left(\dot{J}_a\right)^* = \frac{1}{2}\sum R I_a{}^2 + j\frac{1}{2}\left(\sum \omega L I_a{}^2 - \sum \frac{1}{\omega C} I_a{}^2\right)$$

Note that resistance always absorbs active power (reactive power is equal to zero); inductances and capacitances may contribute to the total reactive power (their active power is equal to zero). Generated complex power is determined similarly to the generated DC power.

RMS current

Formula for average (active) power consists of the amplitudes multiplication, cosine function of the phase difference and the factor 0,5. This factor corresponds to the complex representation through amplitudes. There are several other characteristics of periodical signals (functions):

- peak value – for sinusoidal current peak value is equal to amplitude;
- average value (e.g. average power);
- root mean square (RMS) value;

Average value of the sinusoidal function is equal to zero:

$$I_{av} = \frac{1}{T}\int_0^T i(t)dt = \frac{I_a}{T}\int_0^T \sin(\omega t + \varphi)dt = 0$$

Note that this property may be used as a simple test: does the current (or voltage) have the sinusoidal waveform?

RMS value may be found as square root of the average square of the signal:

$$I_{RMS} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt} = I_a\sqrt{\frac{1}{T}\int_0^T \sin^2(\omega t + \varphi)dt} = \frac{I_a}{\sqrt{2}}$$

This is important formula because it is suggested that engineers use RMS values more frequently than amplitudes or other characteristics. 220 V (in our home) or 220 kV (high-voltage electrical network) – RMS values.

Another reason of the wide span of the RMS values is that, in fact, it *was* much easier to design measurement device that measure RMS than peak value. Nowadays situation is quite different: computerized digital measurement devices are able to indicate amplitude, average and RMS values simultaneously without any difficulties.

As the RMS value is directly proportional to amplitude (with the constant factor) it is possible to use RMS values as basis of the complex representation:

$$\dot{I}_a = I_a e^{j\varphi} \quad \Leftrightarrow \quad \dot{I}_{RMS} = \frac{I_a}{\sqrt{2}} e^{j\varphi}$$

It is just necessary to *choose* proper representation, to *mention* it in the beginning of the solution – “RMS values are used” or “amplitudes are used” and to *remember* about this choice when the natural time-domain representation is obtained.

Complex representation through RMS values allows to rewrite complex power balance in the simple form:

$$\tilde{S} = \dot{V}_{RMS}\left(\dot{I}_{RMS}\right)^*$$

$$\sum \dot{E}_{RMS}\left(\dot{I}_{RMS}\right)^* + \sum \dot{V}_{RMS}\left(\dot{J}_{RMS}\right)^* = \sum RI_{RMS}{}^2 + j\left(\sum \omega LI_{RMS}{}^2 - \sum \frac{1}{\omega C} I_{RMS}{}^2\right)$$

Expressions are more similar to the DC power balance and doesn’t contain additional factor.

General principles and theorems

Complex representation allows to apply all the principles and theorems of the linear circuits that were introduced in the DC analysis chapter:

- superposition principle – the only difference is that all values are complex numbers (partial currents and voltages, coefficients of matrices). Note that it is necessary to refer to mutual partial *impedance* (not mutual partial *resistance*) and mutual partial *admittance* (not mutual partial *conductance*);
- linearity principle – all coefficients are complex numbers;

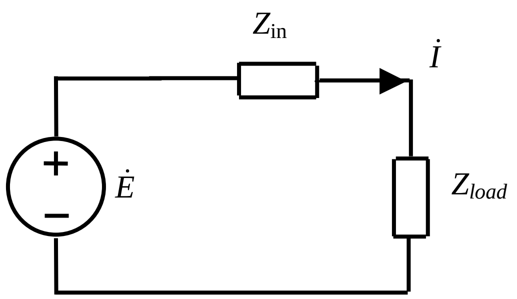
- reciprocity principle – valid for complex values;
- Thevenin's theorem and Norton's theorem – have the same form but it is necessary to use complex open voltage, complex short current and complex input impedance.

Volt-amps diagrams aren't applicable in AC analysis because it is impossible to draw complex function of complex argument on the plane. There are different techniques of graphical representation of the complex functions (impedance, admittance, voltage, current) of the real argument (frequency), but volt-amps characteristics are too difficult to draw and aren't useful in AC analysis.

Matched circuit

Describing DC real power sources we found that it's possible to obtain maximal power transmission from the source to the load when $R_{in}=R_{load}$. There is no point to transmit *complex* power from the source to the load. As the energy flow corresponds to active power it is necessary to transmit *maximal active power*.

Let's describe circuit which consists of the complex e.m.f. source (RMS voltage E), internal impedance Z_{in} and the load impedance Z_{load}.



First of all, it is possible to eliminate total reactive power. Reactive power may be positive or negative, so positive part may be compensated by the negative part. Mathematically it gives the following condition:

$$\operatorname{Im}(Z_{load}) = -\operatorname{Im}(Z_{in})$$

- reactance of the load is equal to internal reactance of the source with opposite sign.

Then situation is similar to the DC analysis – to obtain maximal active power transmission it is necessary to equal real parts of impedances: resistance of the load is equal to the internal resistance of the source.

These statements may be represented by the only expression:

$$Z_{load} = (Z_{in})^*$$

- load impedance is equal to conjugated internal impedance. If the load impedance satisfies this condition, load is called **matched** (with the source). Matching of the generating part of the large scale electrical networks (electrical grids) is one of the most important (and sophisticated!) problems of power engineering. Matching allows to increase efficiency of the power transmission, power plants, consumer equipment. Unmatched networks are non-stable; they have poor *quality* of the signals (non-stable frequency, non-stable (higher or lower) voltage in the network, distortion of the waveform).

Conclusion

Physics gives us precise formulas and reasonable explanations. But we can see that abstract, non-physical formula allows to use very efficient techniques of the power transmission analysis. Reactive power (or compensation of the reactive power) – these words are frequently used by engineers, as frequently as RMS current (or voltage)!

Lecture 8.

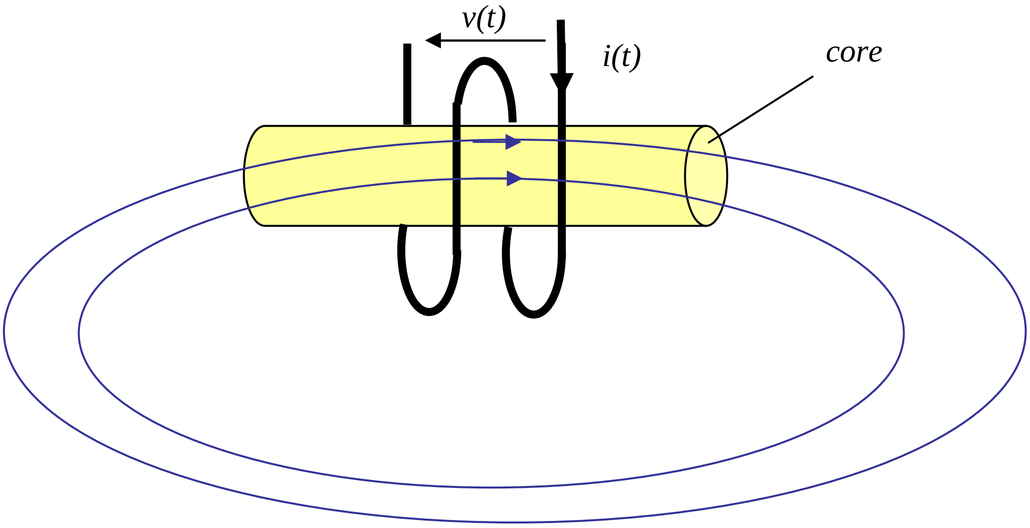
Mutual inductance. Transformer. Ideal transformer.

Introduction

At the beginning of the XX century there was a war in the USA. It is called “currents war” and it was war between the followers of the DC power systems (their leader was Thomas Alva Edison) and the followers of the AC power systems (headed by George Westinghouse). DC motors were simpler, cheaper, more reliable and easier to control than AC motors. DC allows (as AC) to heat something, to recharge batteries (AC needs additional equipment), to use electric bulbs (and there is no flickering light possible when DC system is used). DC is more safe in case of contact with humans (AC is dangerous for heart fibrillation). DC power transmission lines don’t influence all other electrical equipment in the large area (AC lines really do!). There are too many advantages of the DC systems and there is the only advantage of the AC ones. But AC systems had won. Their advantage was weighted heavier than all features of DC systems. This advantage is the transformer – device which allows to change the voltage rate. Transformer makes it possible to transmit power through high-voltage lines (with low current and then with low losses rate) and to use this power in our appliances safely (with relatively low voltage) – and this transformation of the voltage costs very low price (efficiency of the transformers is rather high).

Mutual inductance

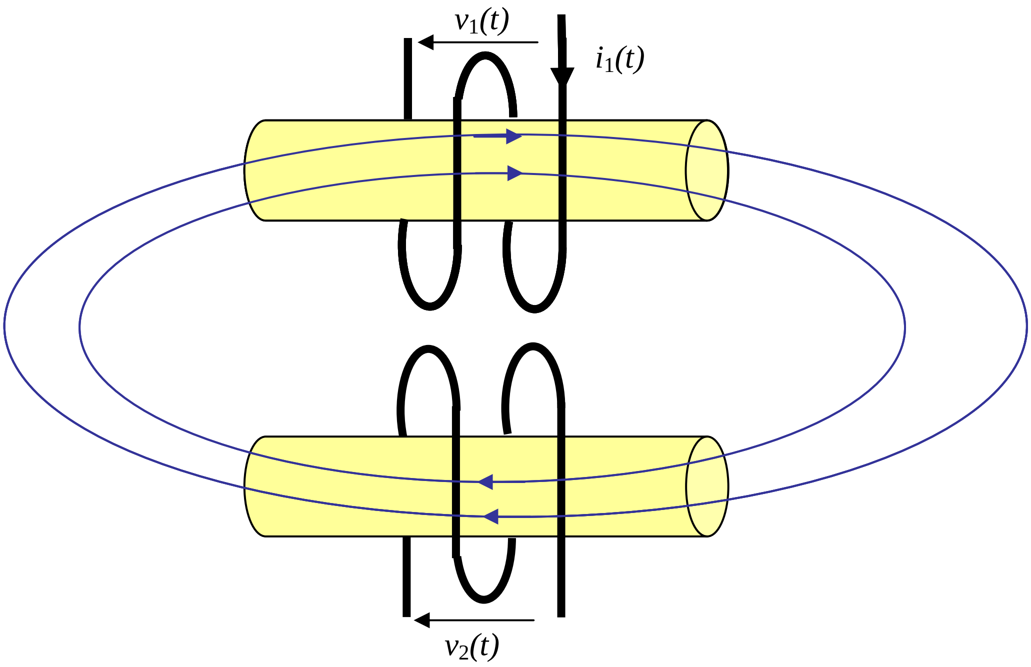
There is very simple logical chain: current induces magnetic field, magnetic field induces voltage. It may be referred as a consequence of the Maxwell’s equations. If the current flows through the turns of the coil, magnetic field is induced mainly inside of the coil (see the following figure). Thus, voltage is induced between the terminals of the coil.



The ratio between magnetic flux linkage and the current is called inductance of the coil or **self inductance**:

$$L = \frac{\Psi}{i} = \frac{w\Phi}{i}$$

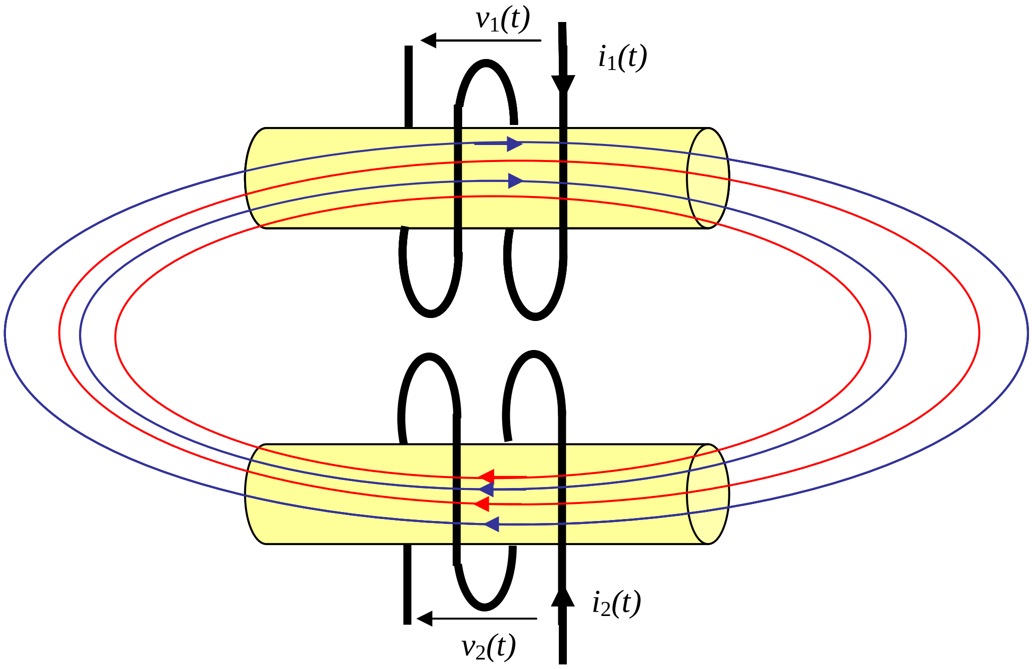
If there are two coils (see figure) and the current flows through one of them, current induce magnetic field which passes through turns of both coils. Therefore, voltage is induced between terminals of both coils. As self inductance is the coefficient between the current and the flux linkage in the first coil, **mutual inductance** is the coefficient between the current and the flux linkage in the second coil:



$$L_1 = \frac{\Psi_1}{i_1} = \frac{w_1\Phi_1}{i_1} \quad M = \frac{\Psi_2}{i_1} = \frac{w_2\Phi_2}{i_1}$$

$$v_1 = \frac{d\Psi_1}{dt} = L_1\frac{di_1}{dt} \quad v_2 = \frac{d\Psi_2}{dt} = M\frac{di_1}{dt}$$

Of course, there may be pair of currents and the full system of equations will have the following form:



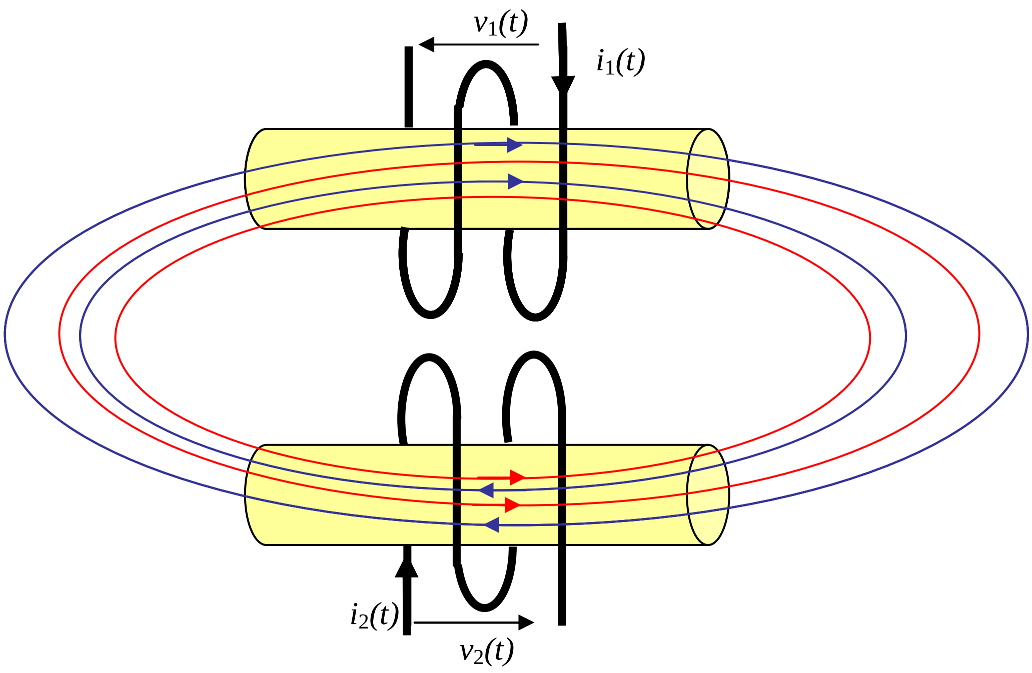
$$L_1 = \frac{\Psi_1}{i_1} = \frac{w_1\Phi_1}{i_1} \quad M_{12} = \frac{\Psi_{12}}{i_2} = \frac{w_1\Phi_{12}}{i_2} \quad L_2 = \frac{\Psi_2}{i_2} = \frac{w_2\Phi_2}{i_2} \quad M_{21} = \frac{\Psi_{21}}{i_1} = \frac{w_2\Phi_{21}}{i_1}$$

$$\begin{cases} v_1 = \frac{d\Psi_1}{dt} + \frac{d\Psi_{12}}{dt} = L_1\frac{di_1}{dt} + M_{12}\frac{di_2}{dt} \\ v_2 = \frac{d\Psi_2}{dt} + \frac{d\Psi_{21}}{dt} = L_2\frac{di_2}{dt} + M_{21}\frac{di_1}{dt} \end{cases}$$

Note that magnetic field which is induced by the first current has the same direction inside the coils as the magnetic field induced by the secondary current.

Reciprocity principle claims that mutual inductances are same $M_{12} = M_{21}$.

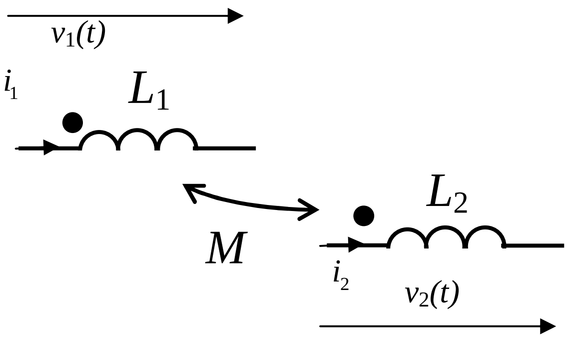
Let's consider case when second current has another direction:

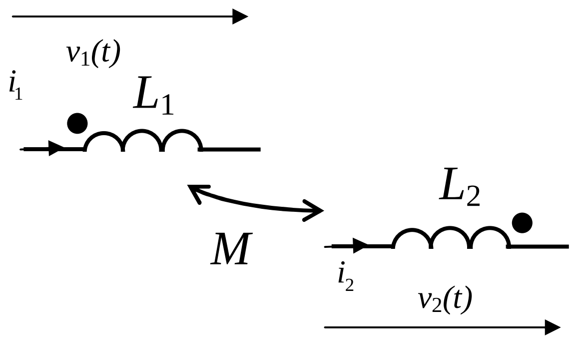


Magnetic fields that are induced by different currents have opposite direction inside the coils, so it is necessary to change the sign of mutual induced voltages:

$$\begin{cases} v_1 = \dfrac{d\Psi_1}{dt} - \dfrac{d\Psi_{12}}{dt} = L_1 \dfrac{di_1}{dt} - M_{12} \dfrac{di_2}{dt} \\ v_2 = \dfrac{d\Psi_2}{dt} - \dfrac{d\Psi_{21}}{dt} = L_2 \dfrac{di_2}{dt} - M_{21} \dfrac{di_1}{dt} \end{cases}$$

We use double-side arrow to show the mutual inductance on the scheme. We use dots to point out *positive* directions of the current. There is simple rule – if both currents flow to the marked (unmarked) terminal mutual induced voltages have same sign as self induced voltages. Otherwise mutual induced voltages have opposite sign. Examples are shown below:





$$\begin{cases} v_1 = L_1 \dfrac{di_1}{dt} + M \dfrac{di_2}{dt} \\ v_2 = L_2 \dfrac{di_2}{dt} + M \dfrac{di_1}{dt} \end{cases}$$

$$\begin{cases} v_1 = L_1 \dfrac{di_1}{dt} - M \dfrac{di_2}{dt} \\ v_2 = L_2 \dfrac{di_2}{dt} - M \dfrac{di_1}{dt} \end{cases}$$

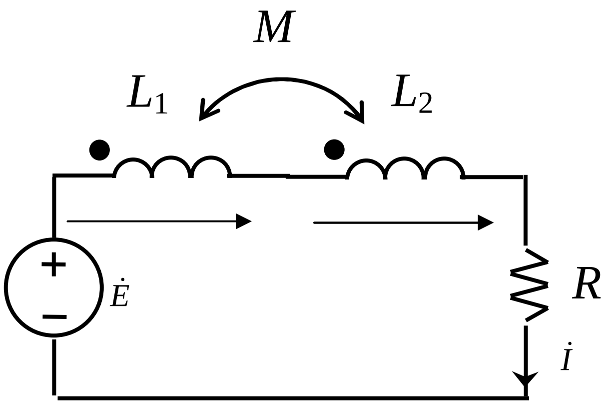
Special coefficient is introduced: **coupling coefficient** connects inductances of the coupled coils with the mutual inductance:

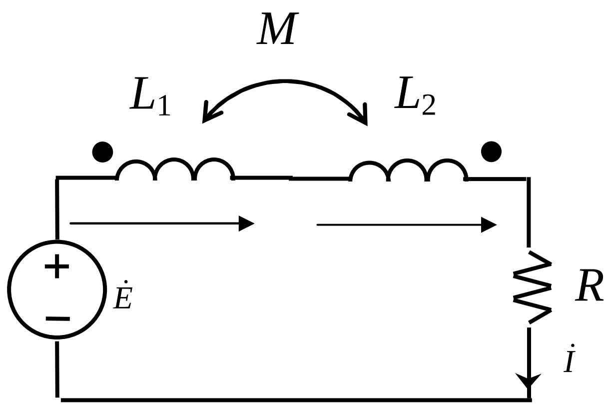
$$M = k\sqrt{L_1 L_2}$$

Coupling coefficient varies from 0 (no magnetic coupling) to 1 (full coupling, there is the same magnetic flux in the both coils). This coefficient is widely used because it shows the rate (or efficiency) of magnetic coupling. Indeed, when it is mentioned that mutual inductance is 5 nH it is incomplete information – it is necessary to know self-inductances (both!) to judge whether the magnetic coupling is effective or is not. But the value of the coupling coefficient of 0,95 gives us enough information to affirm that magnetic coupling is effective.

What is happening with other part of the magnetic flux when coupling coefficient is less than 1? There is **magnetic flux leakage** – the rest part of the magnetic flux spreads throughout except secondary coil.

Let's consider circuit which consists of the resistance and pair of the coupled inductors ("inductor" is the *component* of the circuit, it is an abstract thing and it is modeled by the *element* “inductance” or “self-inductance + mutual inductance” while coil refers to the certain device – it has turns of the wire, core, etc.; it is modeled by one or more elements).





1 mode – current flows into the marked terminals of inductors; KVL leads us to the following expression:

$$\dot{E} = j\omega L_1 \dot{I} + j\omega M\dot{I} + j\omega L_2 \dot{I} + j\omega M\dot{I} + R\dot{I}$$

As there is series connection there is the only current:

$$\dot{E} = \dot{I}\left(j\omega L_1 + j\omega L_2 + 2j\omega M + R\right)$$

Input impedance of the circuit is:

$$Z_{in1} = \frac{\dot{E}}{\dot{I}} = j\omega L_1 + j\omega L_2 + 2j\omega M + R$$

2 mode – current flows into the marked terminals of the first and into unmarked terminal of the second:

$$\dot{E} = j\omega L_1 \dot{I} - j\omega M\dot{I} + j\omega L_2 \dot{I} - j\omega M\dot{I} + R\dot{I}$$

Then, input impedance:

$$Z_{in2} = \frac{\dot{E}}{\dot{I}} = j\omega L_1 + j\omega L_2 - 2j\omega M + R$$

Note that the absolute value of the input impedance is changed dramatically:

$$Z_{in1} - Z_{in2} = 4j\omega M$$

When there is the same e.m.f. source for both modes, current will be greater at second mode. This simple circuit may be used to determine mutual inductance and to mark terminals of the coils *experimentally*.

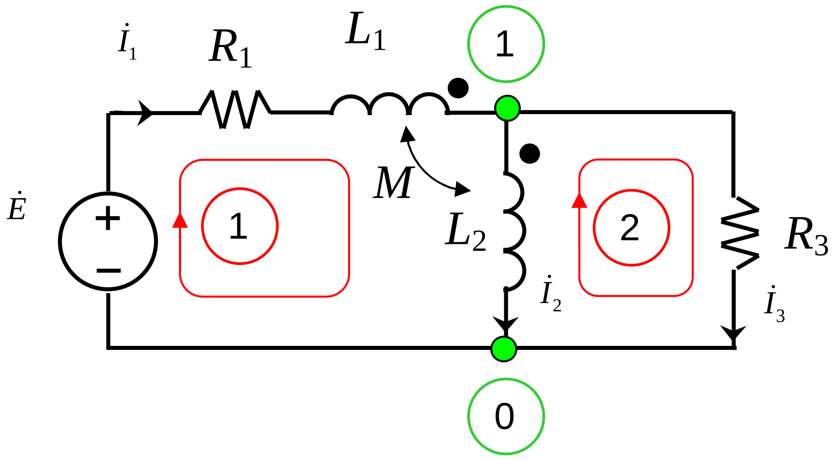
Then, we have to make several remarks:

- self inductance is the important property of all current conductors (wires, cables, stripes on the PCB (printed circuit boards), switchers, semiconductor diodes or transistors, pins of ICs (integrated circuits), etc., not only *coils* (large number of turns of the wire isn't necessary condition to speak about inductance);

- inductance of any conductors has remarkable dependence on the frequency (same thing is truth for the resistance);
- mutual inductance is the important property of all pairs of conductors (pair of the wires, pair of the IC pins, etc.);
- mutual inductance also depends on frequency, but sometimes this dependence isn't important for the analysis;
- sometimes it is necessary to obtain as high coupling coefficient as possible (e.g. transformers require this); sometimes it is necessary to obtain the least possible value of this coefficient (to reduce mutual influence of the communication cables); sometimes it is necessary to obtain the certain value - for example, design of the high-frequency generator.

Circuits with magnetic coupling

This small chapter is a brief overview of the methods of the AC analysis of the circuits with magnetic coupling. First of all, KVL and KCL still give us appropriate number of equations to find all currents in the circuit. It is just necessary to be careful with KVL – remember, that voltage drop on each inductor is the sum of the self-inductance voltage and mutual inductance voltage. As an example, the following circuit is analyzed:



KCL:

$$\dot{I}_1 = \dot{I}_2 + \dot{I}_3$$

KVL:

$$\begin{cases} -\dot{E} + j\omega L_1\dot{I}_1 + R_1\dot{I}_1 - j\omega M\dot{I}_2 + j\omega L_2\dot{I}_2 - j\omega M\dot{I}_1 \\ R_3\dot{I}_3 - \left(j\omega L_2\dot{I}_2 - j\omega M\dot{I}_1\right) = 0 \end{cases}$$

Note, that this approach works with arbitrary number of coupled coils – just sum all mutual voltages.

As KCL and KVL are applicable, it is possible to use matrix formulation. The only difference is that branch impedance matrix won't be diagonal:

$$Z = \begin{bmatrix} R_1 + j\omega L_1 & -j\omega M & 0 \\ -j\omega M & j\omega L_2 & 0 \\ 0 & 0 & R_3 \end{bmatrix}$$

Note that branch admittance matrix should be computed as:

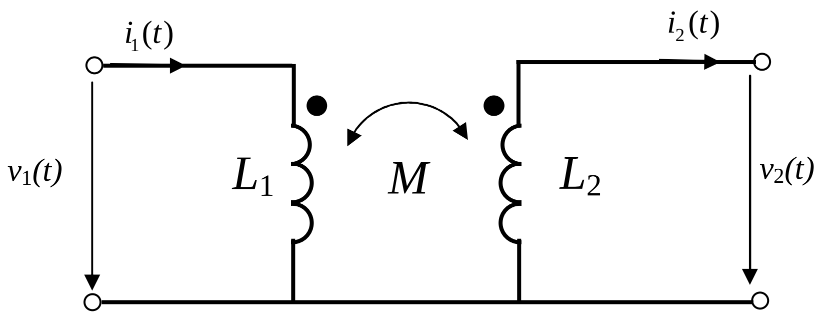
$$Y = Z^{-1}$$

Thus, we are able to use nodal analysis and mesh analysis as usually. But the necessity of the matrix inversion makes it non-convenient to use nodal analysis in the *symbolic* form.

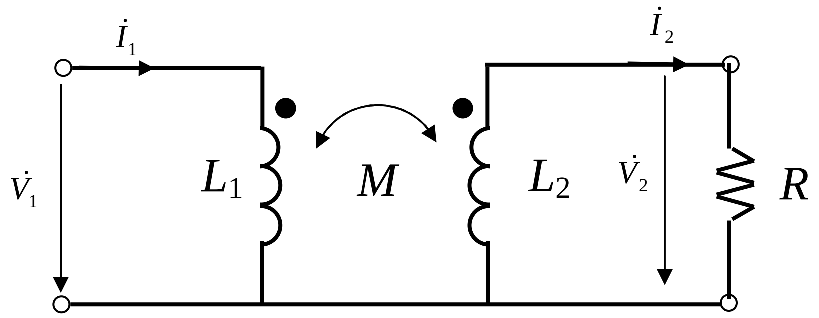
Transformer

First of all, transformer is a complicated device which uses magnetic coupling to increase or decrease the voltage (or current) on the output in comparison to the input voltage (current). Real transformer (especially for high-power applications) may have large sizes (very high voltage transformers are really huge – each transformer weights more than 10000000 kg); there is a core inside it (core consists of thousands of thin steel plates); coils and core are placed in the room filled with special oil or gas. Realistic model of the transformer requires coupled solution of the Maxwell's equations, hydrodynamic equations, mechanical equations, thermodynamic equations. Simplified but still realistic equivalent scheme (electrical model) of the transformer includes large number of elements: self and mutual inductances of the turns (for each turn separately!), capacitances between turns, resistances of the turns, additional resistances that model power dissipation inside the core and etc.

The simple model of the transformer is not very realistic; it consists of two inductors with magnetic coupling (so, there are three elements here – 2 self-inductances and 1 mutual inductance):



Input of the transformer is formed by the pair of **primary terminals** – terminals of the **primary winding**. Output is the pair of **secondary terminals** of the **secondary winding**. One can use KCL and KVL to analyze circuit with such model of the transformer. The important property of the transformer is that its input impedance depends on the load of the secondary winding. Let's consider the following circuit:



KVL gives us:

$$\begin{cases} \dot{V}_1 = j\omega L_1 \dot{I}_1 - j\omega M \dot{I}_2 \\ j\omega L_2 \dot{I}_2 - j\omega M \dot{I}_1 + R\dot{I}_2 = 0 \end{cases}$$

So, input impedance is:

$$Z_{IN} = \frac{\dot{V}_1}{\dot{I}_1} = j\omega L_1 + \frac{\omega^2 M^2}{j\omega L_2 + R}$$

The second part of this expression is called **induced impedance**.

Ideal transformer

For non-precise estimation of the circuit performance the simplest model of the transformer may be used. It is called **ideal transformer** and this model has the only parameter – **turns ratio** *n*. Turns ratio is the ratio of the number of turns of the secondary winding to the numbers of turns of the primary winding:

$$n = \frac{w_2}{w_1}$$

In this model it is assumed that there is no leakage flux, and, correspondingly, there is the same magnetic flux in the every turn of the every winding. Thus, primary voltage is proportional to the number of turns of the primary winding and secondary voltage – to the number of turns of the secondary winding. Therefore, ratio of the secondary (output) voltage to primary (input) voltage is equal to the turns ratio:

$$\Phi_1 = \Phi_2 = \Phi \quad \rightarrow \quad \Psi_1 = w_1\Phi, \quad \Psi_2 = w_2\Phi$$

$$\begin{cases} v_1 = \dfrac{d\Psi_1}{dt} = w_1 \dfrac{d\Phi}{dt} \\ v_2 = \dfrac{d\Psi_2}{dt} = w_2 \dfrac{d\Phi}{dt} \end{cases} \rightarrow \quad \frac{v_2}{v_1} = \frac{w_2}{w_1} = n$$

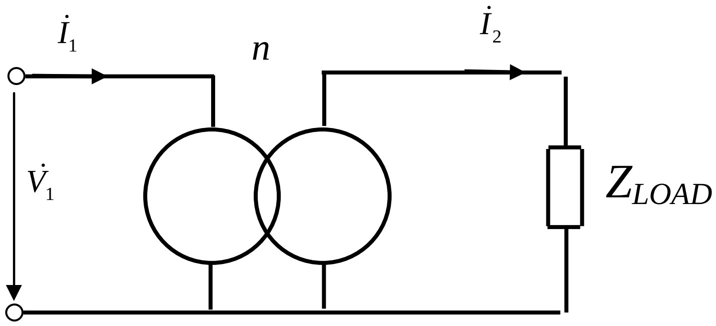
In other hand, magnetic flux is proportional to “ampere-turns” of each winding (magnetic field is proportional to the current and to the number of turns). Then, as there is same magnetic flux, one can equal ampere-turns of primary and secondary winding:

$$\Phi_1 = \Phi_2 = \Phi \quad \rightarrow \quad w_1 i_1 = \Phi, \quad w_2 i_2 = \Phi \quad \rightarrow \quad \frac{i_2}{i_1} = \frac{w_1}{w_2} = n^{-1}$$

So, ratio of the secondary (output) current to the primary (input) current is proportional to n^{-1}.
Note that the input power is equal to the output power of the ideal transformer:

$$p_{IN} = v_1 i_1 = \frac{v_2}{n} i_2 n = v_2 i_2 = p_{OUT}$$

Another important note is that induced impedance (input impedance of the ideal transformer) is equal to the load impedance divided by the square of the turns ratio:



$$Z_{IN} = \frac{\dot{V}_1}{\dot{I}_1} = \frac{\dot{V}_2}{n\,\dot{I}_2 n} = \frac{\dot{V}_2}{n^2 \dot{I}_2} = \frac{Z_{LOAD}}{n^2}$$

This property may be used for the matching of the load with the real power source. If there is internal resistance R_{in} of the model of the real source and there is load resistance R_{load} and it is impossible (by any reason) to change them, it is necessary to use transformer that has turns ratio as:

$$n = \sqrt{\frac{R_{LOAD}}{R_{IN}}}$$

Indeed:

$$Z_{INPUT} = \frac{R_{LOAD}}{n^2} = R_{IN}$$

that means that maximal active power is transferred from the source to the load.
Note that it must be AC source to use transformers – there is no induced DC voltage on the coil!

Conclusion

Magnetic coupling allows us to build different devices, first of all, transformers. There is just a little difference (and there are no serious difficulties) in AC analysis of the circuits with the magnetic coupling. We’ve studied different models of the transformers and there is the important difference between the ideal transformer and all other models – ideal transformer doesn't dissipate power (active or reactive). We always use simplest model first (to estimate general performance of the device) and use more sophisticated model then (to determine many details of the design).

Lecture 9.

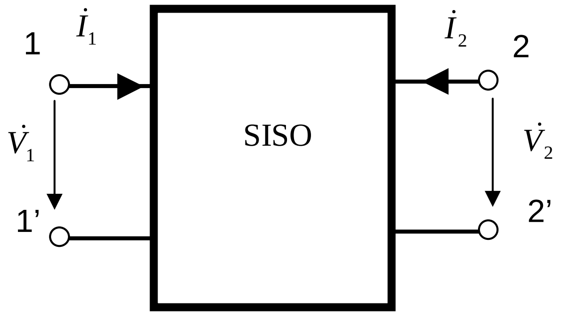
Two-ports systems. Two-ports parameters.

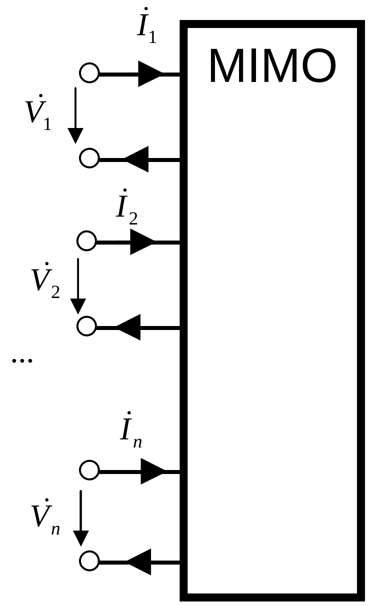
Introduction

Power engineering deals with the sources and receptors of the energy. Source is the power plant or another "big" object; receptors are factories, plants, etc. – rather "big". Electrical grid that connects them is spanned through regions and countries and is, in fact, stationary system. Electronics, computer sciences, control theory and "low-power" electrical engineering deal with relatively small devices that process data, transform signals, etc. Such devices may be connected with others, different signal sources may be attached (antennas, sensors, coders, etc.) to them and it isn't known *a priory* what will be the environment of this particular device. This situation requires to have certain information about all devices that is necessary to join. Of course, there is no need to know all the details, internal structure, schemes of these devices. The only thing that is important is the description of the input, output and the transfer mechanism (it may be transfer of information or energy or specific signal) from the input to the output. This lecture describes simple AC models of the systems that have two-ports – input port and output port.

SISO and MIMO systems

First of all, **port** is the pair of the terminals of the circuit. One and the same terminal of the circuit may be used to form several ports (it is common terminal of these ports). "Port" is just a shorter name for the pair of terminals and, of course, this term doesn't mean that these terminals have unique properties or special symbols. One can consider *any* pair of terminals as a port. "Input port" or just "input" usually means that signal come to this port from elsewhere. "Output port" or "output" usually means that signal come out of this port. Words "input" and "output" assume that our circuit (device, system) has *specific direction* of the electrical energy transfer. Our first task is a kind of *identification* of the circuit by using as simple set of parameters as possible. From this point of view, to make a full description of the circuit (device, system) it is necessary to *test* each port as input and as output. In other words we should *excite* each port and measure responses on all other ports. So that, it is necessary to omit terms "input" and "output" sometimes and use numbers of ports (e.g. "primary and secondary ports", or "fifth port" without mentioning is it input port or output).





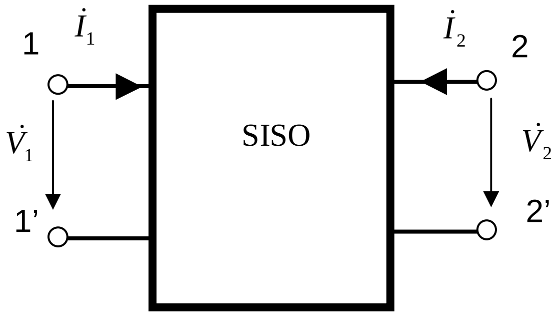
Generally speaking, the common case is the system (circuit) with large number of ports (part of them may be referred as inputs, rest of them – as outputs) – multi-port system or **MIMO** system (multiple input, multiple output). The most important particular case is the **SISO** system (single input, single output) or **two-ports system**.

SISO system allows us to use terms "primary terminals" and "secondary terminals" or "input" and "output" to discriminate ports.

Two-ports parameters

Port signal of the system may be described by the pair of functions – port voltage and port current. AC analysis uses complex representation of sinusoidal voltages and currents. Port voltage is the voltage between the port terminals, port current flows into the circuit through one port terminal and returns through other (see the following figure). Two-ports system has pair of the port voltages and pair of the port currents. Therefore, simple AC model of the system is an algebraic link between this quadruplet of complex values. One can use different representations of this system and each representation gives quadruplet of the complex coefficients – **two-ports parameters**.

For example, let's connect pair of the voltages with the pair of the currents:



$$\begin{cases} \dot{V}_1 = Z_{11}\dot{I}_1 + Z_{12}\dot{I}_2 \\ \dot{V}_2 = Z_{21}\dot{I}_1 + Z_{22}\dot{I}_2 \end{cases}$$

It is the system of the two linear algebraic equations with complex coefficients. As all coefficients have the same dimension (Ohms) and this dimension corresponds to the resistance or impedance, this set of complex parameters is called Z-parameters or **impedance parameters**. This system of algebraic equations may be represented in a matrix form:

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix}$$

Note that any multi-port system may be represented same way, so impedance parameters are quite general:

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \\ \dot{V}_4 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \dot{I}_3 \\ \dot{I}_4 \end{bmatrix}$$

Similarly, it is possible to find the dependence between the pair of the currents and the pair of the voltages:

$$\begin{cases} \dot{I}_1 = Y_{11}\dot{V}_1 + Y_{12}\dot{V}_2 \\ \dot{I}_2 = Y_{21}\dot{V}_1 + Y_{22}\dot{V}_2 \end{cases}$$

All four complex coefficients are measured in Siemens (as admittances), so this set of parameters is called Y-parameters or **admittance parameters**.

Matrix form simply shows that Y-matrix is an inverse Z-matrix:

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^{-1}$$

So, one can conclude that admittance and impedance parameters have very similar properties.

There is another possible set of parameters which connects primary voltage and secondary current with the primary current and the secondary voltage:

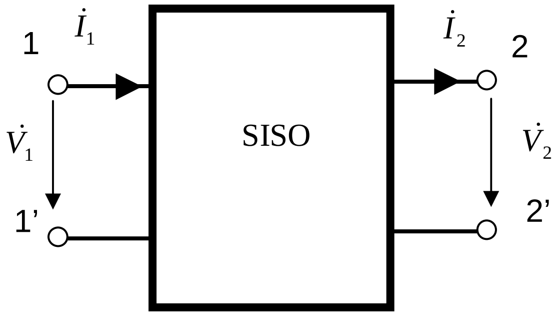
$$\begin{cases} \dot{V}_1 = H_{11}\dot{I}_1 + H_{12}\dot{V}_2 \\ \dot{I}_2 = H_{21}\dot{I}_1 + H_{22}\dot{V}_2 \end{cases}$$

Here H_{11} is measured in Ohms, H_{22} is measure in Siemens, H_{12} and H_{21} are dimensionless. Different dimension of the coefficients allows to call this set of parameters as **hybrid parameters** or H-parameters. Hybrid parameters are widely used in electronics to specify linear mode of the bipolar transistor.

Note that hybrid parameters are introduced for *SISO systems only*.

There are many other sets of parameters: scattering (S) parameters that are used mainly at very high frequencies, modified hybrid parameters, etc. And there is one another set of parameters which is widely used: **transmission parameters** (or ABCD-parameters).

Transmission parameters are introduced differently (in comparison with Z, Y or H) – ports are unequal: primary port is described as input, secondary port – as output (see the following figure).



Then, transmission parameters link input voltage and input current with output ones:

$$\begin{cases} \dot{V}_1 = A\dot{V}_2 + B\dot{I}_2 \\ \dot{I}_1 = C\dot{V}_2 + D\dot{I}_2 \end{cases}$$

As hybrid parameters, H-parameters have different dimensions – *A* and *D* are dimensionless, *B* is measured in Ohms, *C* is measured in Siemens. And as hybrid parameters, ABCD parameters are introduced for *SISO systems only*.

Identification of SISO system

In this particular case, identification means determination of the two-ports parameters. It is shown latter, that all parameters set are connected together and there is no need to determine every (or particular) parameter set – the most convenient set should be used under the certain conditions. In fact, it is possible to use very simple approach to find ABCD-parameters:

1) connect source (theoretically it may be e.m.f. or current source, only real source is available for experiment) to the primary terminals (input);

2) connect *open* circuit to the secondary terminals (output), measure primary voltage and primary current, measure secondary voltage (secondary current is equal to zero as an open circuit is attached);

3) compute pair of coefficients *A* and *C*:

$$A = \left.\frac{\dot{V}_1}{\dot{V}_2}\right|_{\dot{I}_2=0}$$

$$C = \left.\frac{\dot{I}_1}{\dot{V}_2}\right|_{\dot{I}_2=0}$$

4) connect *short* circuit to the secondary terminals (output), measure primary voltage and primary current, measure secondary current (secondary voltage is equal to zero as a short circuit is attached);

5) compute pair of coefficients B and D:

$$B = \left.\frac{\dot{V}_1}{\dot{I}_2}\right|_{\dot{V}_2=0}$$

$$D = \left.\frac{\dot{I}_1}{\dot{I}_2}\right|_{\dot{V}_2=0}$$

Similar approach may be used for other parameters: it is useful to attach open circuit to primary and secondary terminals to determine impedance parameters:

$$Z_{11} = \left.\frac{\dot{V}_1}{\dot{I}_1}\right|_{\dot{I}_2=0}$$

$$Z_{21} = \left.\frac{\dot{V}_2}{\dot{I}_1}\right|_{\dot{I}_2=0}$$

$$Z_{12} = \left.\frac{\dot{V}_1}{\dot{I}_2}\right|_{\dot{I}_1=0}$$

$$Z_{22} = \left.\frac{\dot{V}_2}{\dot{I}_2}\right|_{\dot{I}_1=0}$$

short circuits are useful for admittance parameters:

$$Y_{11} = \left.\frac{\dot{I}_1}{\dot{V}_1}\right|_{\dot{V}_2=0}$$

$$Y_{21} = \left.\frac{\dot{I}_2}{\dot{V}_1}\right|_{\dot{V}_2=0}$$

$$Y_{12} = \left.\frac{\dot{I}_1}{\dot{V}_2}\right|_{\dot{V}_1=0}$$

$$Y_{22} = \left.\frac{\dot{I}_2}{\dot{V}_2}\right|_{\dot{V}_1=0}$$

hybrid parameters require short circuit on primary terminals and open circuit on secondary:

$$H_{11} = \left.\frac{\dot{V}_1}{\dot{I}_1}\right|_{\dot{V}_2=0}$$

$$H_{21} = \left.\frac{\dot{I}_2}{\dot{I}_1}\right|_{\dot{V}_2=0}$$

$$H_{12} = \left.\frac{\dot{V}_1}{\dot{V}_2}\right|_{\dot{I}_1=0}$$

$$H_{22} = \left.\frac{\dot{I}_2}{\dot{V}_2}\right|_{\dot{I}_1=0}$$

Same approach is used for the computational determination of the two-ports parameters.

Note that some of the parameters have relatively clear physical meaning – e.g. hybrid parameter H_{11} is the input impedance of the circuit while there is open circuit on secondary terminals and H_{22} is the output admittance of the circuit while there is short circuit on primary terminals.

It is necessary to underline that there are two practically important properties of the linear SISO system: *reciprocity* and *symmetry*.

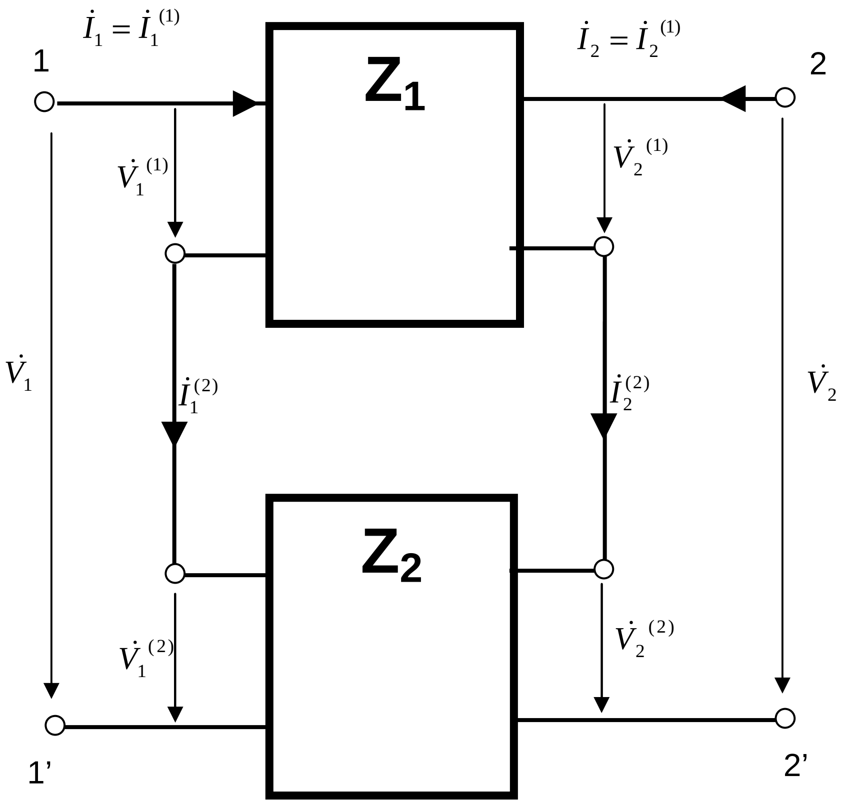
Reciprocal system has $Z_{12} = Z_{21}$, $Y_{12} = Y_{21}$, $H_{12} = -H_{21}$ and $AD-BC=1$.

Symmetrical system is always reciprocal and, in addition, $Z_{11} = Z_{22}$, $Y_{11} = Y_{22}$, and $A=D$. Note that symmetry isn't shown by the hybrid parameters as clear as by others. Anyway, symmetrical system has two independent parameters of any type. In other words, symmetrical system may be represented by the symmetrical matrix (of certain type – ***Z*** or ***Y***).

Connections of SISO systems

There are several connections available for SISO systems: *series*, *parallel*, *series-parallel*, *parallel-series* and *cascade*.

If there is same primary current and same secondary current of two SISO systems, there is **<u>series connection</u>** of the SISO systems:

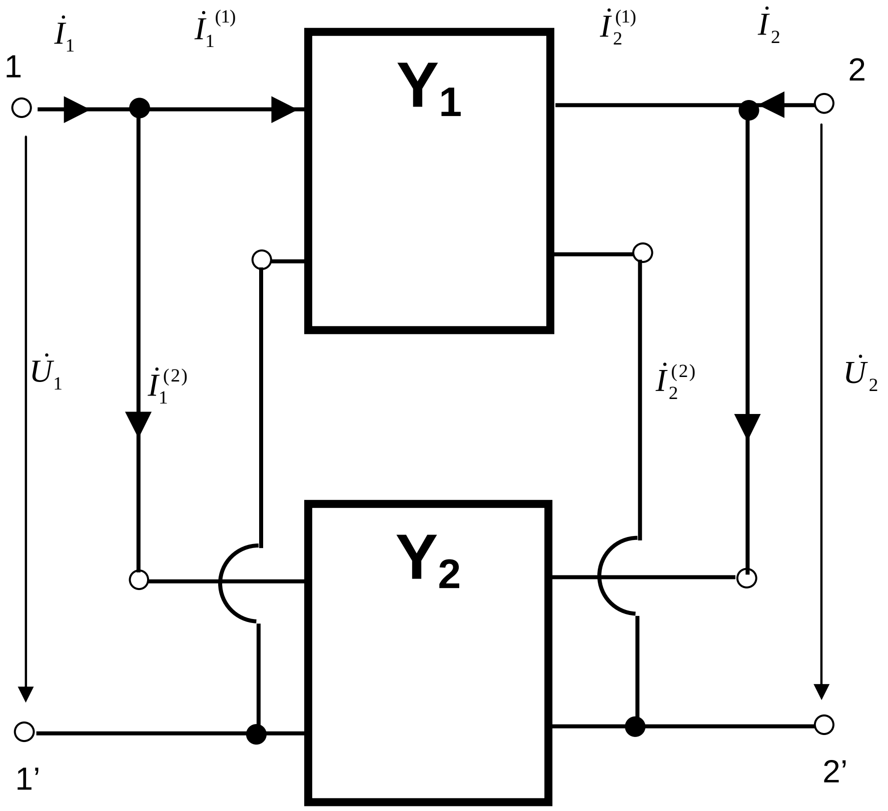


In this case common primary voltage is the sum of primary voltages; common secondary voltage is equal to the sum of secondary voltages. Thus, series connection of two SISO systems may be represented by the equivalent SISO system and matrix of the impedance parameters of the equivalent system is the sum of impedance matrices of the original systems:

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 = \begin{bmatrix} Z_{11}^{(1)} & Z_{12}^{(1)} \\ Z_{21}^{(1)} & Z_{22}^{(1)} \end{bmatrix} + \begin{bmatrix} Z_{11}^{(2)} & Z_{12}^{(2)} \\ Z_{21}^{(2)} & Z_{22}^{(2)} \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} Z_{11}^{(1)} + Z_{11}^{(2)} & Z_{12}^{(1)} + Z_{12}^{(2)} \\ Z_{21}^{(1)} + Z_{21}^{(2)} & Z_{22}^{(1)} + Z_{22}^{(2)} \end{bmatrix}$$

If there is same primary voltage and same secondary voltage of two SISO systems, there is **parallel connection** of the SISO systems:

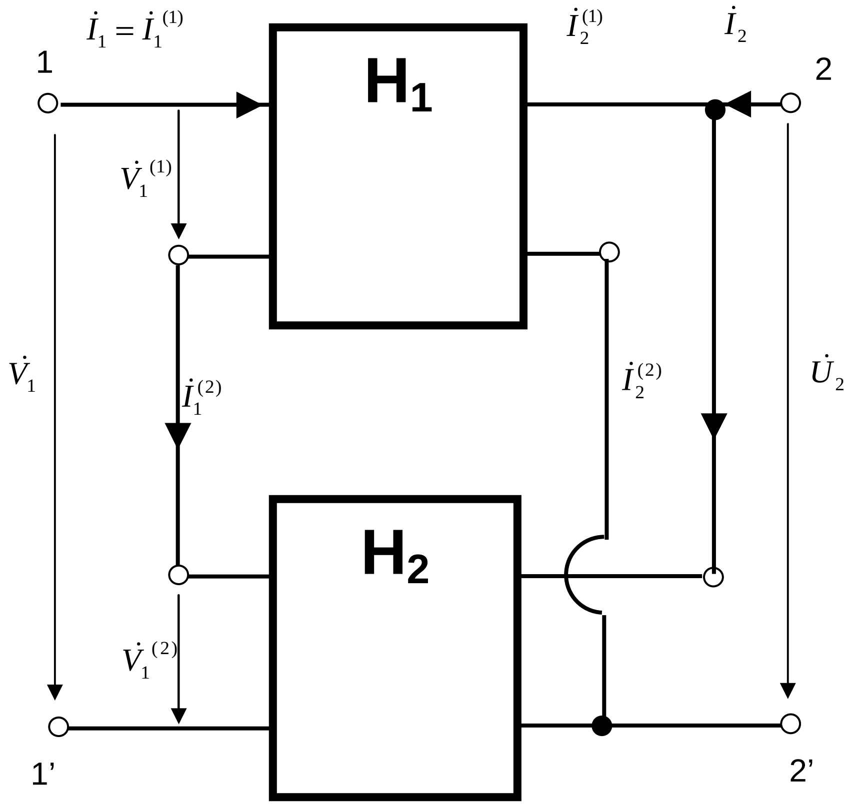


In this case common primary current is the sum of primary currents; common secondary current is equal to the sum of secondary currents. Thus, parallel connection of two SISO systems may be represented by the equivalent SISO system and matrix of the admittance parameters of the equivalent system is the sum of admittance matrices of the original systems:

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = \begin{bmatrix} Y_{11}^{(1)} & Y_{12}^{(1)} \\ Y_{21}^{(1)} & Y_{22}^{(1)} \end{bmatrix} + \begin{bmatrix} Y_{11}^{(2)} & Y_{12}^{(2)} \\ Y_{21}^{(2)} & Y_{22}^{(2)} \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} Y_{11}^{(1)} + Y_{11}^{(2)} & Y_{12}^{(1)} + Y_{12}^{(2)} \\ Y_{21}^{(1)} + Y_{21}^{(2)} & Y_{22}^{(1)} + Y_{22}^{(2)} \end{bmatrix}$$

Series-parallel connection combines series connection of primary terminals and parallel connection of secondary terminals. Equivalent SISO system has hybrid matrix as the sum of the original hybrid matrices:

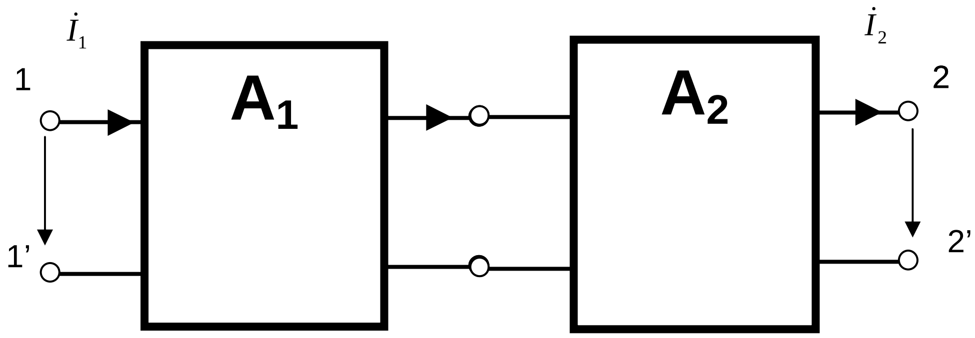


$$H = H_1 + H_2 = \begin{bmatrix} H_{11}^{(1)} & H_{12}^{(1)} \\ H_{21}^{(1)} & H_{22}^{(1)} \end{bmatrix} + \begin{bmatrix} H_{11}^{(2)} & H_{12}^{(2)} \\ H_{21}^{(2)} & H_{22}^{(2)} \end{bmatrix}$$

$$H = \begin{bmatrix} H_{11}^{(1)} + H_{11}^{(2)} & H_{12}^{(1)} + H_{12}^{(2)} \\ H_{21}^{(1)} + H_{21}^{(2)} & H_{22}^{(1)} + H_{22}^{(2)} \end{bmatrix}$$

Parallel-series connection combines parallel connection of primary terminals and series connection of secondary terminals. Equivalent SISO system has modified hybrid matrix as the sum of the original modified hybrid matrices (we didn't introduce them, someone, may be, will derive it at home).

Cascade connection (it is available for SISO systems only) is an attachment of the input of one system to the output of another:



Cascade connection of SISO systems may be represented by the equivalent SISO system and the matrix of the transmission parameters of the equivalent system is a product of the original ABCD matrices:

$$A = A_1 + A_2 = \begin{bmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & D^{(1)} \end{bmatrix} \begin{bmatrix} A^{(2)} & B^{(2)} \\ C^{(2)} & D^{(2)} \end{bmatrix}$$

$$A = \begin{bmatrix} A^{(1)}A^{(2)} + B^{(1)}C^{(2)} & A^{(1)}B^{(2)} + B^{(1)}D^{(2)} \\ C^{(1)}A^{(2)} + D^{(1)}C^{(2)} & C^{(1)}B^{(2)} + D^{(1)}D^{(2)} \end{bmatrix}$$

Cascade connection is the most widely used connection of the two-ports systems because of simple reason: data (signal, energy) processing is the sequence of the different operations (e.g. coding, modulation, amplification, transmission, amplification again, demodulation, decoding – this sequence forms simple radio transmission system). Each operation is processed in the separate SISO system and the sequence of the operations corresponds to the cascade connection of the systems.

Conclusion

AC analysis of the two-ports system is relatively simple – it leads to algebraic equations. As always, complex numbers are used to represent currents and voltages of the same frequency. So it is necessary to obtain two-ports parameters at each frequency if there are signals with different frequencies.

We've considered one side of the problem – how to build the model of the system (or how to identify system). Another side is how to use these parameters for the further analysis.

Lecture 10.

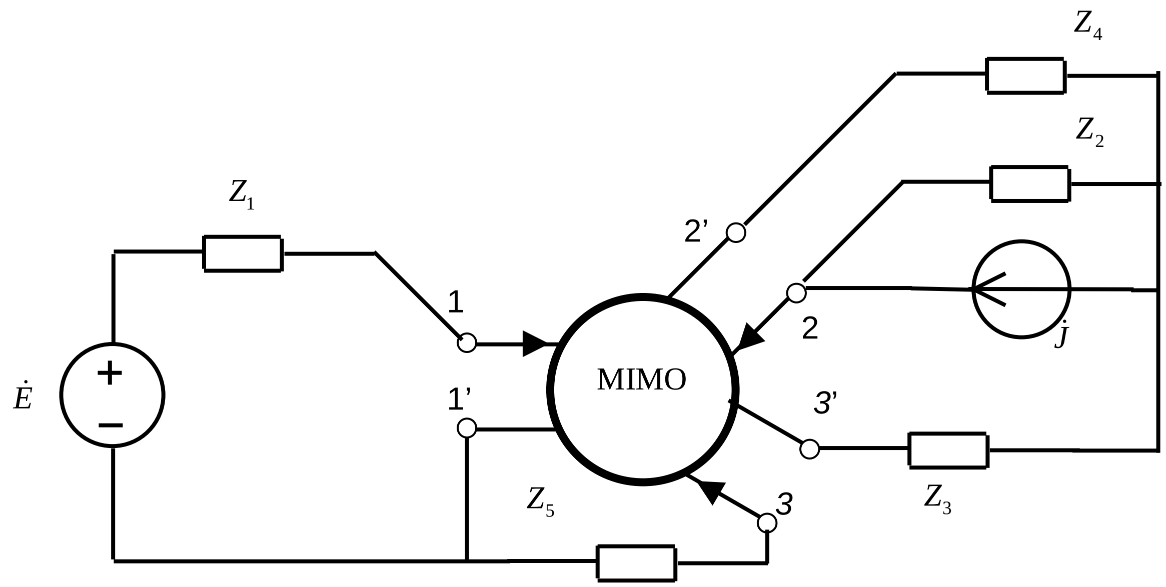
Controlled sources. Schemes of two-ports systems. Operational amplifier.

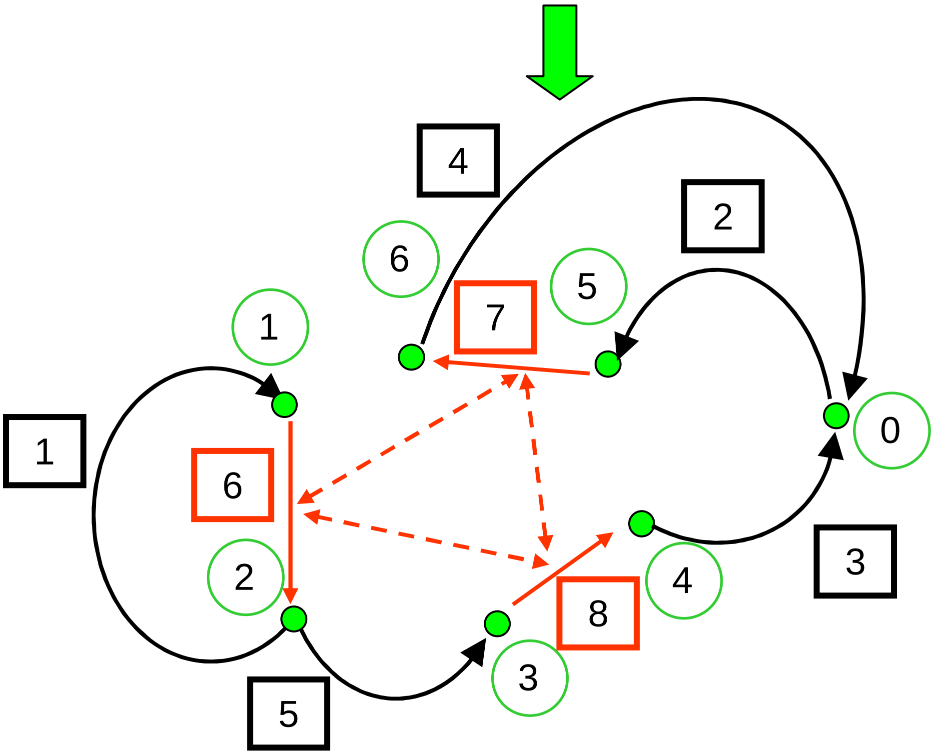
Introduction

This is, in fact, second part of the previous lecture. If system is identified and two-ports parameters are known it is necessary to "build-in" this system into another, large scale system (circuit). Straight way of the analysis is based on the ordinary KVL+KCL matrix formulation. One can mention the disadvantage of this method – it isn't very obvious representation, it is rather abstract. Another way is that the SISO system interior should be substituted by the certain equivalent scheme that depends on the parameters set. The basic element of this equivalent scheme is a controlled source.

Matrix solution

This is general method that is based on KCL and KVL matrix formulation. Each terminal of the SISO or MIMO system is represented as a node in the circuit. And it is assumed that there is a number of links between nodes (this formulation seems to be similar to the matrix analysis of the circuits with the magnetic coupling). Each internal link corresponds to the certain parameter (Z or Y-parameters may be used):





$$\dot{\mathbf{E}} = \begin{bmatrix} \dot{E} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \dot{\mathbf{J}} = \begin{bmatrix} 0 \\ -\dot{J} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then, the matrix of the Z-parameters of the system is inserted into the impedance matrix of the general branches (or matrix of Y-parameters is inserted into the admittance matrix of the general branches):

$$\mathbf{Z} = \begin{bmatrix} Z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Z_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_{11} & Z_{12} & Z_{13} \\ 0 & 0 & 0 & 0 & 0 & Z_{21} & Z_{22} & Z_{23} \\ 0 & 0 & 0 & 0 & 0 & Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Further solution is similar to the ordinary case, nodal analysis or mesh analysis may be used.

$$\mathbf{AYA}^T\dot{\boldsymbol{\varphi}} = -\mathbf{AY\dot{E}} + \mathbf{A\dot{J}}$$

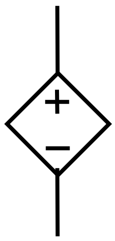
$$\mathbf{Y} = \mathbf{Z}^{-1}$$

It is necessary to mention one serious problem that occurs in relatively rare particular cases: it is possible, that matrix of impedance parameters is singular, so that admittance parameters aren’t defined for the system; in this case nodal analysis (which uses admittance matrix) is unavailable. Similarly, it is possible to have singular matrix of the admittance parameters, thus impedance matrix isn’t defined and mesh analysis is unavailable. There are several special methods that expand nodal analysis (or mesh analysis) to make it possible to analyze circuits with singular matrices. These methods are out of scope of our course.

Controlled sources

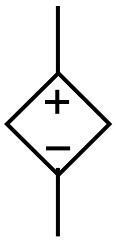
Before we start analysis by using equivalent circuits it is necessary to introduce new elements – **controlled sources**. Ordinary sources (e.m.f. source and current source) may be called independent sources in contrast to controlled sources. Note that we are describing *elements* or *ideal models*, not the *real* sources.

There are four possible controlled sources in the *linear* circuit:



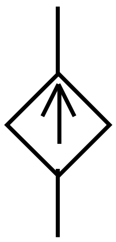
$$\dot{E}(\dot{V}) = k\dot{V}$$

e.m.f. source controlled by the voltage



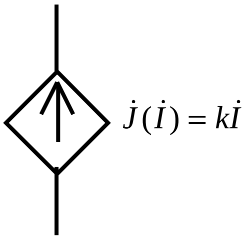
$$\dot{E}(\dot{I}) = Z\dot{I}$$

e.m.f. source controlled by the current



$$\dot{J}(\dot{V}) = Y\dot{V}$$

current source controlled by the voltage



current source controlled by the current

These special symbols are used in US, in Russia we use same symbols for controlled and independent sources.

Non-linear circuit may consist of the controlled sources with non-linear dependences (e.g. exponential, logarithmical).

Each linear controlled source has only one parameter – *complex coefficient* that links this source with the controlling variable (voltage or current). It may be dimensionless coefficient (between e.m.f. and voltage or between currents – *voltage gain* and *current gain*), or it may be measured in Ohms (this coefficient is called *transresistance*, it links controlling current and e.m.f.) or in Siemens (*transconductance,* links controlling voltage and the current of the source).

Then, it is necessary to modify one important definition – while computing input impedance it is necessary to set all *independent* sources to zero (make circuit passive) but all *controlled* sources should stay turned on! That means that it is necessary to apply input voltage and compute input current. It isn't possible to transform circuit using just a parallel or series connections when there are controlled sources in the circuit.

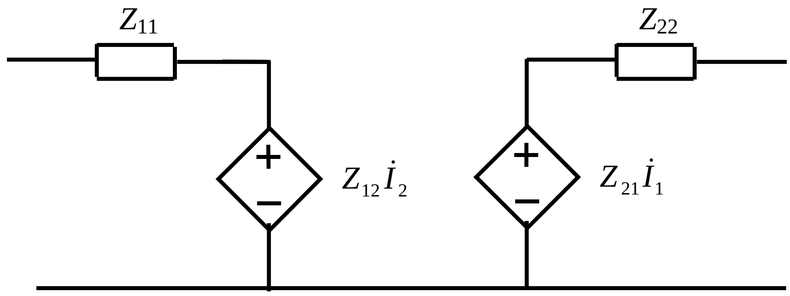
Another important note is that it is necessary to add *equations* for *controlling variables* into the total system of equation for the circuit with controlled sources.

Equivalent schemes of two-ports systems

Each set of parameters leads to specific equivalent scheme. Each scheme consists of two passive elements and two controlled sources.

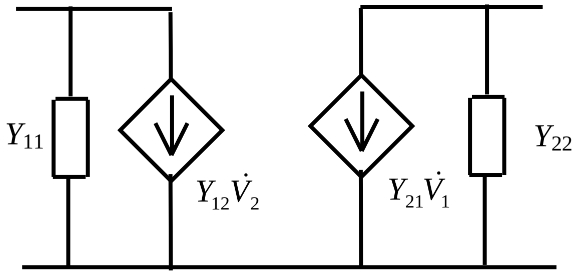
For impedance parameters:

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix}$$



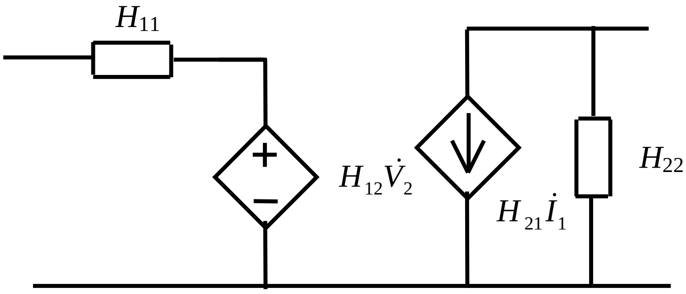
For admittance parameters:

$$\begin{cases} \dot{I}_1 = Y_{11}\dot{V}_1 + Y_{12}\dot{V}_2 \\ \dot{I}_2 = Y_{21}\dot{V}_1 + Y_{22}\dot{V}_2 \end{cases}$$



For hybrid parameters:

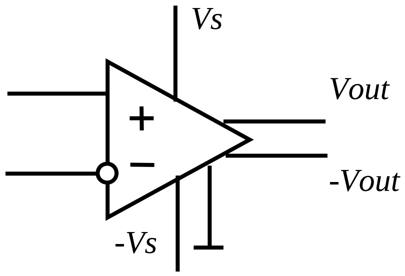
$$\begin{cases} \dot{V}_1 = H_{11}\dot{I}_1 + H_{12}\dot{V}_2 \\ \dot{I}_2 = H_{21}\dot{I}_1 + H_{22}\dot{V}_2 \end{cases}$$



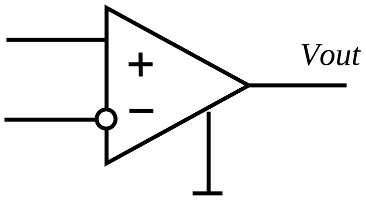
Unfortunately, there is no equivalent scheme that uses transmission parameters.

Operational amplifier

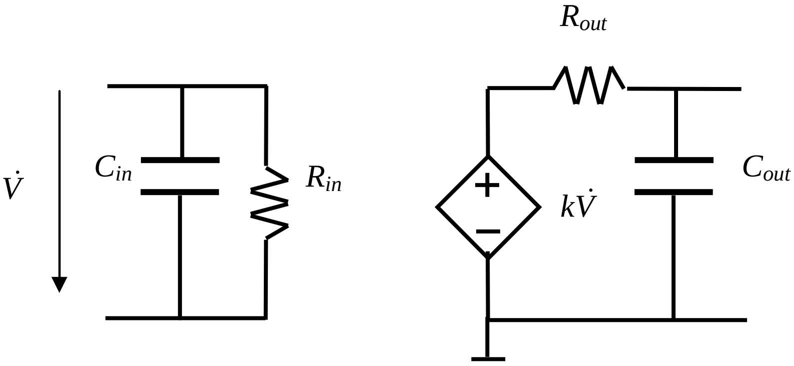
We use ideal elements – controlled sources to build a model of the SISO (or MIMO) system. But there is real device that behaves as controlled source. It is **operational amplifier** (*OpAmp*) – electronic (semiconductor) device that is widely and differently used. Usually operational amplifiers are combined in integrated circuits. Each amplifier has, at list, 6 terminals – power supply (positive DC voltage) "*+Vs"*; power supply (negative DC voltage) "*–Vs"*; ground (zero potential) – "*gnd"*; positive (forward) input "+"; negative (*inverse*) input "–"; output voltage "*+Vout*"; inverse output voltage "*–Vout*" (optional).



AC analysis of the linear mode of the OpAmp requires to determine signals (voltages) of both input terminals and then, output voltage may be computed. So, equivalent scheme of the linear model contains only 4 terminals – inputs, output and ground:



Output voltage (between output terminal and ground) of the amplifier is equal to the voltage difference on the input terminals (positive minus negative) that is multiplied by the coefficient k – voltage gain of the OpAmp; there are input and output impedances of the OpAmp:



Typical amplifier of low frequency range (less than 100MHz) has $k > 10^6$, $R_{in} > 10^6\ \Omega$, $C_{in} < 1$ nF, $R_{out} < 10\ \Omega$, $C_{out} < 1$ nF; its price is about \$1 per unit (integrated circuit). Very high frequency amplifier (greater than 10 GHz) has $k > 20$, $R_{in} \approx 50\ \Omega$, $C_{in} < 1$ pF, $R_{out} \approx 50\ \Omega$, $C_{out} < 1$ pF; its price is higher than \$100 per unit (integrated circuit).

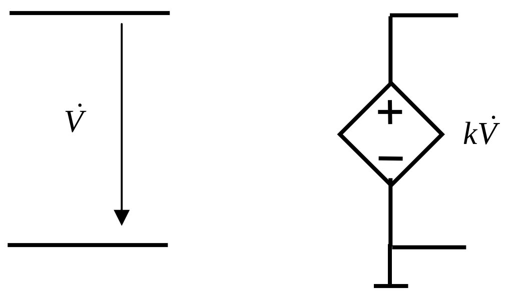
One can find, for example, impedance parameters of the operational amplifier:

$$\begin{bmatrix} \dfrac{1}{g_{in} + j\omega C_{in}} & 0 \\ \dfrac{k}{g_{in} + j\omega C_{in}} & \dfrac{1}{g_{iout} + j\omega C_{out}} \end{bmatrix}$$

Ideal operational amplifier

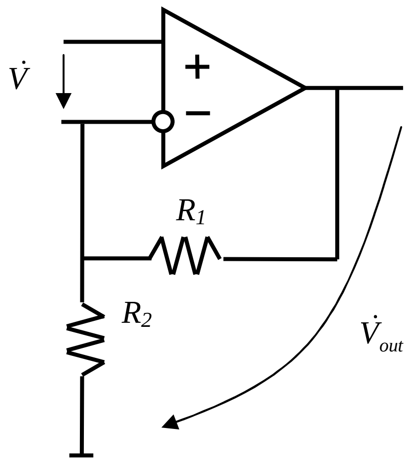
There is a kind of uncertainty in the parameters of the amplifier – gain is given as just a lower bound of the uncertain range. In fact, it doesn't matter practically, because amplifiers aren't used as simply – they are "circled" by the several components (resistors or capacitors) that stabilize voltage gain, improve effective frequency range, enable additional frequency dependences.

Thus, simplified model of the operational amplifier may be used – **ideal operational amplifier**. Ideal OpAmp has infinite voltage gain, infinite input resistance, zero output resistance and zero capacitances. Equivalent scheme is shown below:

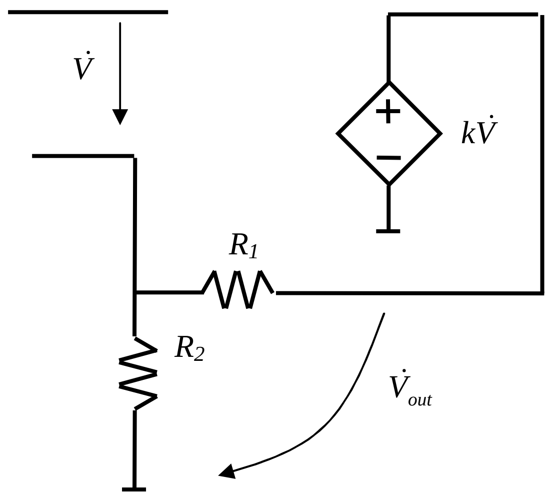


Of course, if there is finite (non-zero) voltage difference on input terminals, there is infinite output voltage – this model cannot be used without additional *external* (with respect to OpAmp) elements.

Let's consider circuit with ideal OpAmp and pair of resistors:



Resistors form **negative feedback loop**, because they connect output with the inverse input. Inserting equivalent scheme of the ideal OpAmp one can obtain:



KVL gives us potential of the inverse input:

$$\dot{\varphi}_{-} = \frac{\dot{V}_{out} R_2}{R_1 + R_2} = \frac{k\dot{V} R_2}{R_1 + R_2} = \frac{k(\dot{\varphi}_{+} - \dot{\varphi}_{-}) R_2}{R_1 + R_2}$$

$$\dot{\varphi}_{-} = \frac{\frac{kR_2}{R_1 + R_2} \dot{\varphi}_{+}}{1 + \frac{kR_2}{R_1 + R_2}}$$

Input voltage difference is:

$$\dot{V} = \dot{\varphi}_{+} - \dot{\varphi}_{-} = \dot{\varphi}_{+} \left(1 - \frac{\frac{kR_2}{R_1 + R_2}}{1 + \frac{kR_2}{R_1 + R_2}} \right) = \dot{\varphi}_{+} \left(\frac{1}{1 + \frac{kR_2}{R_1 + R_2}} \right) = \dot{\varphi}_{+} \left(\frac{R_1 + R_2}{R_1 + R_2 + kR_2} \right)$$

And as voltage gain k approaches infinity:

$$\lim_{k \to \infty} (\dot{V}) = 0$$

Then, it is possible to find the output voltage as a limit:

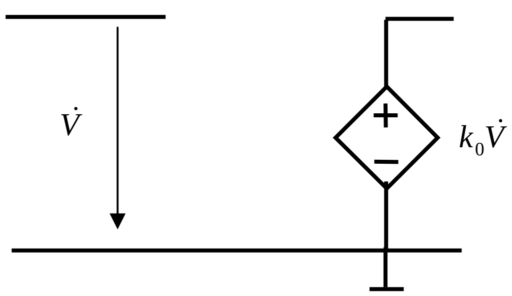
$$\dot{V}_{out} = k(\dot{\varphi}_{+} - \dot{\varphi}_{-}) = k\dot{\varphi}_{+} \left(\frac{R_1 + R_2}{R_1 + R_2 + kR_2} \right)$$

$$\lim_{k \to \infty} (\dot{V}_{out}) = \left(\frac{R_1 + R_2}{R_2} \right) \dot{\varphi}_{+} = \left(1 + \frac{R_1}{R_2} \right) \dot{\varphi}_{+} = k_0 \dot{\varphi}_{+}$$

Note that output voltage is directly proportional to the potential of the positive input of the OpAmp and new equivalent voltage gain is:

$$k_0 = 1 + \frac{R_1}{R_2}$$

Firstly, it is possible to use simple equivalent scheme (this scheme consists of the only element – controlled source) for the circuit with OpAmp and negative feedback loop.



Secondly, it is possible to tune voltage gain changing one of the resistances. Range of the voltage gain is from 1 ($R_2 \to \infty$) to ∞ ($R_2 = 0$).

Conclusion

This lecture completes formal AC analysis of two-ports systems (and particularly – multi-port systems). Every representation is useful under the certain conditions, but it is much more important to get basic knowledge of the principles of system analysis, of system identification. These principles are used not only in an electrical engineering, but in electronics, computer sciences, control theory and etc.

Lecture 11.

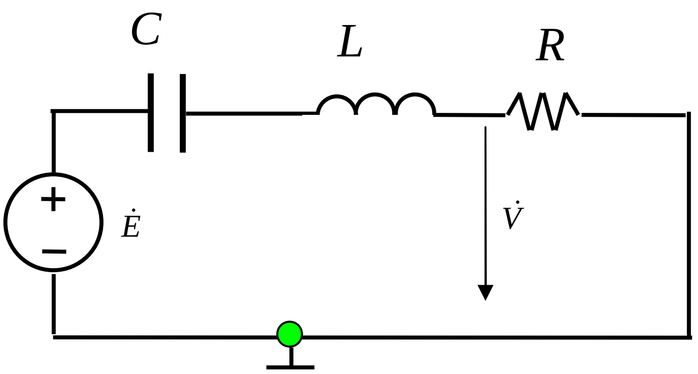
Resonance. Frequency response. Transfer function.

Introduction

At the beginning of the AC analysis chapter we mentioned that complex impedance of inductance (or capacitance) depends on the frequency. This means that *response* (whatever it is) computed (measured) at one frequency isn't valid at another frequency. There are different useful effects that are connected with such dependence. And there is a certain technique that expands AC analysis to cover frequency range.

Resonance

Let's consider RLC loop – classic example of many interesting phenomena. This circuit consists of the series connection of the e.m.f. source, resistance, capacitance and inductance:



Current is computed by using KVL:

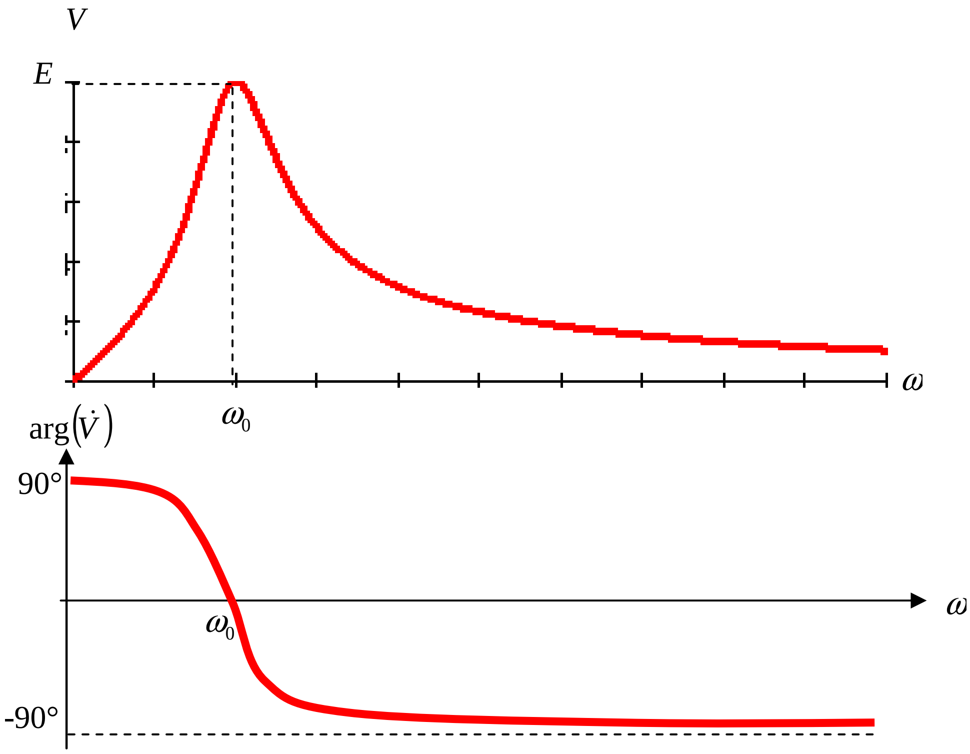
$$-\dot{E} + j\omega L\dot{I} + \frac{1}{j\omega C}\dot{I} + R\dot{I} = 0$$

$$\dot{I} = \frac{\dot{E}}{j\omega L + \frac{1}{j\omega C} + R}$$

Voltage drop on the resistance is evaluated by the Ohm's law:

$$\dot{V} = R\dot{I} = \frac{\dot{E}R}{j\omega L + \frac{1}{j\omega C} + R}$$

Changing frequency of the source it is possible to obtain plot of the amplitude of the voltage drop on the resistance and plot of the phase of the same voltage:



There is an interesting point (angular frequency is ω_0, frequency – f_0) at which amplitude of the voltage reaches maximal value (this value is equal to the electromotive force). Phase of the voltage is equal to zero at this point (of course, phase of the current is also equal to zero).

Our formulas show that this specific frequency is equal to:

$$\dot{V}_{\max} = \dot{V}(\omega_0) = \frac{\dot{E}R}{R} = \dot{E} = E\angle 0°$$

$$j\omega_0 L + \frac{1}{j\omega_0 C} = 0 \quad \rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

There is the point at which reactance of the capacitor is equal to reactance of the inductance. Therefore, complex impedance of the capacitance compensates complex impedance of the inductance.

Indeed, input impedance of the circuit is:

$$Z_{IN} = j\omega L + \frac{1}{j\omega C} + R$$

At point ω_0 input impedance is equal to R. This means that input power source will produce active power only:

$$\tilde{S}(\omega_0) = \dot{E}\left[\dot{I}(\omega_0)\right]^* = \frac{E^2}{R} = P(\omega_0)$$

Indeed complex power balance leads to:

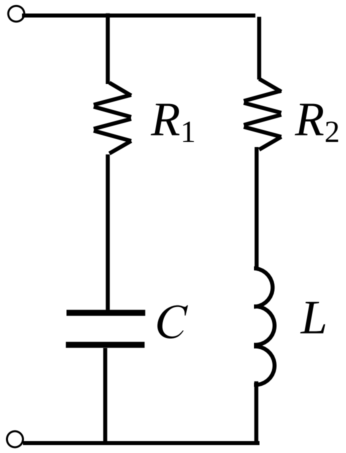
$$Q = \omega L I^2 - \frac{1}{\omega C} I^2 \quad \rightarrow \quad Q(\omega_0) = I^2 \left(\omega_0 L - \frac{1}{\omega_0 C} \right) = 0$$

At ω_0 complex power of capacitance has the same value and opposite sign as complex power of inductance. Physically we consider capacitance as an element that stores energy in the electric field while inductance stores energy in the magnetic field. Equality of reactive power corresponds to the equality of the magnetic and electric field energy. This phenomenon (equality of the energy of the electric field to the energy of the magnetic field) is called **resonance**. Frequency ω_0 (or f_0) is called **resonance frequency**.

In the circuit theory resonance may be defined as phenomenon that occurs in the *passive* circuit that consists of *inductances and capacitances* when the input *reactance* (or input *susceptance*) is equal to zero.

In fact, this definition of the resonance shows that resonance frequencies and even presence of the resonance phenomena depend on the choice of input terminals of the circuit. There is a class of the circuits, for example, that have resonance at each frequency point! Of course, another situation is possible – when circuit consists of inductance and capacitance but doesn’t have resonance frequencies at all.

Let’s consider following circuit:



Input admittance may be found as:

$$Y_{IN} = \frac{1}{R_1 + \frac{1}{j\omega C}} + \frac{1}{R_2 + j\omega L} = \frac{R_1 - \frac{1}{j\omega C}}{R_1^{\ 2} + \frac{1}{\omega^2 C^2}} + \frac{R_2 - j\omega L}{R_2^{\ 2} + j\omega^2 L^2}$$

Then, it is necessary to locate imaginary part:

$$\mathrm{Im}(Y_{IN}) = \frac{\frac{1}{\omega C}}{R_1^2 + \frac{1}{\omega^2 C^2}} + \frac{-\omega L}{R_2^2 + j\omega^2 L^2}$$

Resonance frequency may be determined as:

$$\mathrm{Im}(Y_{IN}(\omega_0)) = 0 \quad \rightarrow \quad \frac{\frac{1}{\omega_0 C}}{R_1^2 + \frac{1}{\omega_0^2 C^2}} + \frac{-\omega_0 L}{R_2^2 + j\omega_0^2 L^2} = 0$$

$$-R_1^2 \omega_0 L - \frac{L}{\omega_0 C^2} + R_2^2 \frac{1}{\omega_0 C} + \frac{\omega_0 L^2}{C} = 0$$

$$-R_1^2 \omega_0^2 L C^2 - L + R_2^2 C + \omega_0^2 L^2 C = 0$$

$$\omega_0 = \sqrt{\frac{L - R_2^2 C}{L^2 C - R_1^2 L C^2}}$$

when

$$\frac{L - R_2^2 C}{L^2 C - R_1^2 L C^2} > 0 \quad \rightarrow \quad \begin{cases} R_2 < \sqrt{\frac{L}{C}} \quad and \quad R_1 < \sqrt{\frac{L}{C}} \\ or \\ R_2 > \sqrt{\frac{L}{C}} \quad and \quad R_1 > \sqrt{\frac{L}{C}} \end{cases}$$

or there are no resonance frequencies when

$$\frac{L - R_2^2 C}{L^2 C - R_1^2 L C^2} < 0 \quad \rightarrow \quad \begin{cases} R_2 < \sqrt{\frac{L}{C}} \quad and \quad R_1 > \sqrt{\frac{L}{C}} \\ or \\ R_2 > \sqrt{\frac{L}{C}} \quad and \quad R_1 < \sqrt{\frac{L}{C}} \end{cases}$$

or there is resonance at each frequency when

$$\frac{L - R_2^2 C}{L^2 C - R_1^2 L C^2} = 0 \quad \rightarrow \quad R_2 = \sqrt{\frac{L}{C}} \quad and \quad R_1 = \sqrt{\frac{L}{C}}$$

Resonance phenomenon plays an important role in the synthesis (design) of selective devices (filters), radio engineering, analysis of the high power transformers, etc.

Quality factor

There is a special parameter introduced in the electrodynamics for the resonance systems – quality factor Q. It is defined as a ratio of the stored energy (in the electric or in the magnetic form) to the power losses in the system at rsosnance frequency. In the circuit theory quality factor is a ratio of reactive power on the inductance (or capacitance) to the active power that is absorbed into resistance (or, may be, resistances).

In the RLC loop, at the resonance frequency, quality factor is:

$$Q = \frac{I^2 R}{\omega_0 L I^2} = \frac{R}{\omega_0 L} = R\sqrt{\frac{C}{L}}$$

Note that the ratio of the amplitude of the voltage on the inductance (capacitance) to the amplitude of the voltage on the resistance gives the same quality factor:

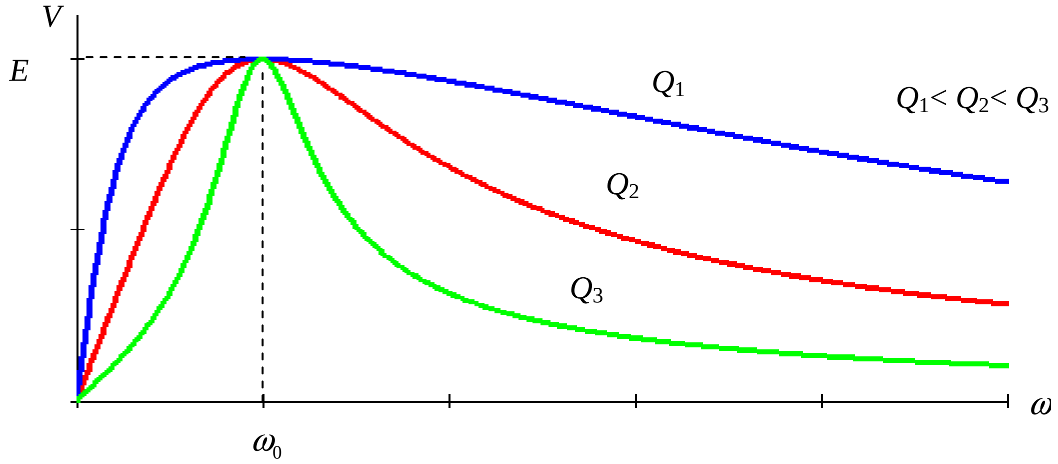
$$Q = \frac{IR}{\omega_0 L I} = \frac{R}{\omega_0 L} = R\sqrt{\frac{C}{L}}$$

$$Q = \frac{IR}{\frac{1}{\omega_0 C} I} = \frac{R}{\frac{1}{\omega_0 C}} = R\sqrt{\frac{C}{L}}$$

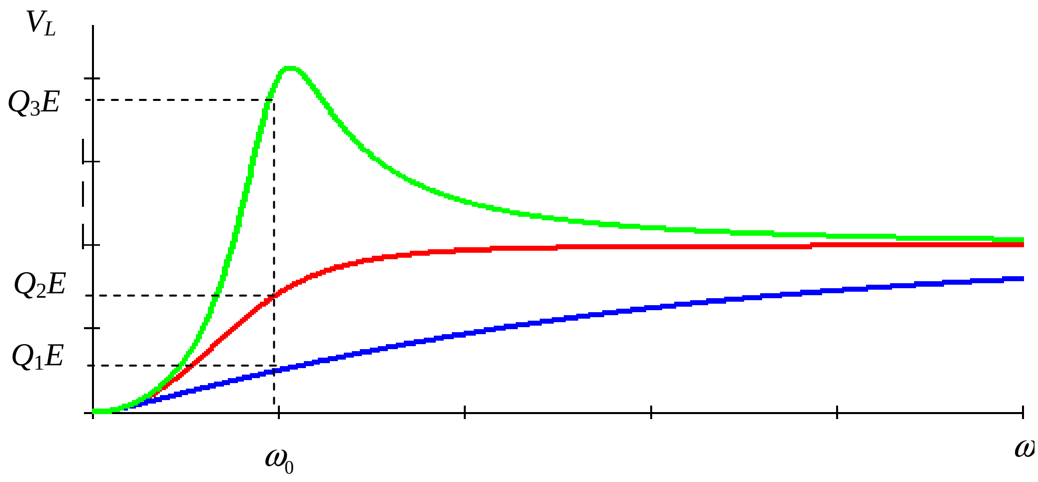
One can rewrite expression for the voltage drop on the resistance in other form:

$$\dot{V} = \frac{\dot{E}R}{j\omega L + \frac{1}{j\omega C} + R} = \frac{j\omega C\dot{E}R}{(j\omega)^2 LC + 1 + j\omega CR} = \frac{j\omega\dot{E}\frac{R}{L}}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = \frac{j\omega\dot{E}\frac{\omega_0}{Q}}{(j\omega)^2 + j\omega\frac{\omega_0}{Q} + \omega_0^2}$$

This form allows to analyze influence of the quality factor on the shape of the resonance curve:



Note that the narrow curve corresponds to higher quality factor. Voltage drops on the inductance and voltage drop on the capacitance behave relatively similar:

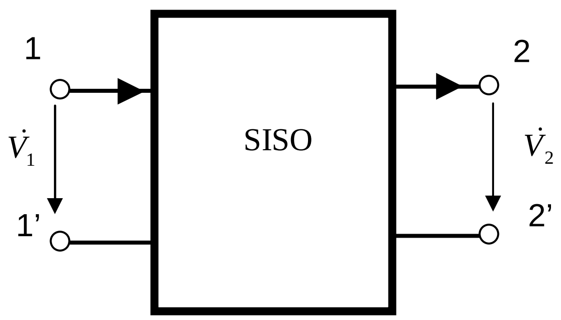


Important note: relatively similar curves may be obtained in the circuits without resonance phenomena – for example, circuits with resistances, capacitances and operational amplifier (but without inductances) may produce same shape of the frequency dependencies of the voltages.

Transfer function

As it was shown above frequency dependencies of the voltages and currents are connected with the choice of the input terminals of the circuit. Now we'll turn back to the SISO systems that have definitely specified input and output terminals. And, as two-ports parameters are used for AC analysis at single frequency point, there is special representation for the circuit reaction, caused by the input signal with an arbitrary frequency. This representation is called **transfer function**.

Transfer function is a complex function of frequency:



$$W(j\omega) = \frac{\dot{V}_2(j\omega)}{\dot{V}_1(j\omega)}$$

Usually, argument is pointed as $j\omega$ to underline complex character of the transfer function.

Transfer function may be represented in *canonic* form:

$$W(j\omega) = \frac{(j\omega)^m b_m + (j\omega)^{m-1} b_{m-1} + \ldots + (j\omega)^2 b_2 + j\omega b_1 + b_0}{(j\omega)^n + (j\omega)^{n-1} a_{n-1} + \ldots + (j\omega)^2 a_2 + j\omega a_1 + a_0}$$

There is a polynomial function of the complex argument *j*ω in the numerator and another polynomial function in the denominator. It is important, that coefficients of the polynomial functions are real positive values. The *order* of the transfer function corresponds to the order (highest power) of the *denominator* – *n*. It may be shown, that the order of numerator cannot be higher than order of the denominator ($m < n$).

Frequency response

Transfer function of a SISO system may be represented graphically. Graphical representation of the transfer function is called **frequency response**. Of course, it is impossible to plot a graph of the transfer function, because of its complex character. There are five possible types of the frequency response representation:

$W(\omega) = |W(j\omega)|$

- **magnitude response** – shows dependence of the magnitude (absolute value) of the transfer function on frequency;

$\varphi(\omega) = \arg(W(j\omega))$

- **phase response** – shows dependence of the argument (phase angle) of the transfer function on frequency;

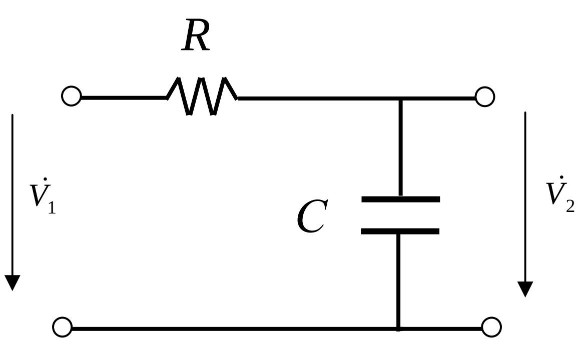
$\mathrm{Re}[W(j\omega)]$

- **real frequency response** – shows dependence of the real part of the transfer function on frequency;

$\mathrm{Im}[W(j\omega)]$

- **imaginary frequency response** – shows dependence of the imaginary part of the transfer function on frequency;
- **frequency response locus** – transfer function is represented as a curve on the complex plane (frequency axis is not available).

Let’s consider simple RC circuit:

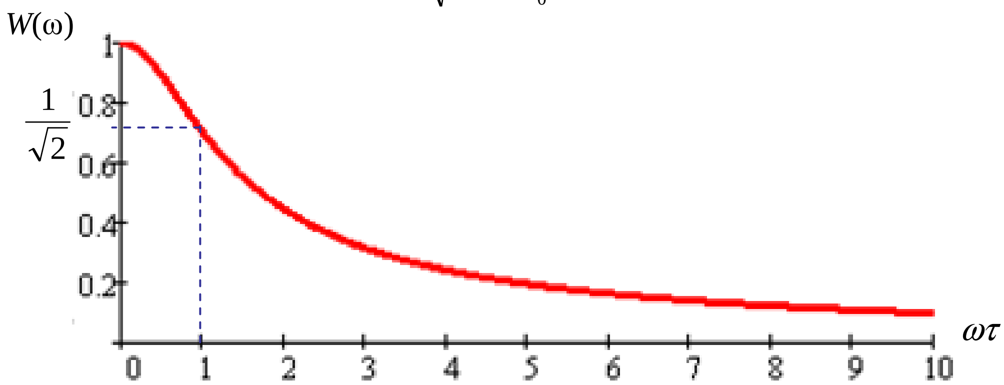


There is a transfer function:

$$W(j\omega)=\frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C}+R}=\frac{1}{1+j\omega CR}=\frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}=\frac{b_0}{j\omega+a_0}=\frac{\frac{1}{\tau}}{j\omega+\frac{1}{\tau}}$$

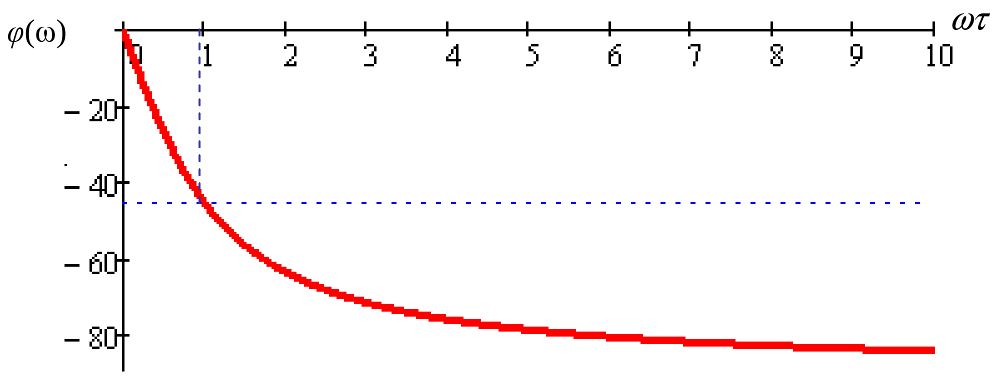
Magnitude response:

$$W(\omega)=\left|\frac{b_0}{j\omega+a_0}\right|=\frac{|b_0|}{|j\omega+a_0|}=\frac{b_0}{\sqrt{\omega^2+a_0{}^2}}$$

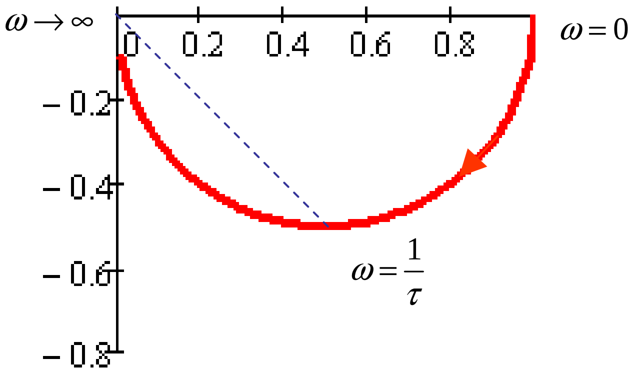


Phase response:

$$\varphi(\omega)=\arg\left(\frac{b_0}{j\omega+a_0}\right)=\arg(b_0)-\arg(j\omega+a_0)=0°-\arctan\left(\frac{\omega}{a_0}\right)=-\arctan(\omega\tau)$$



Locus:

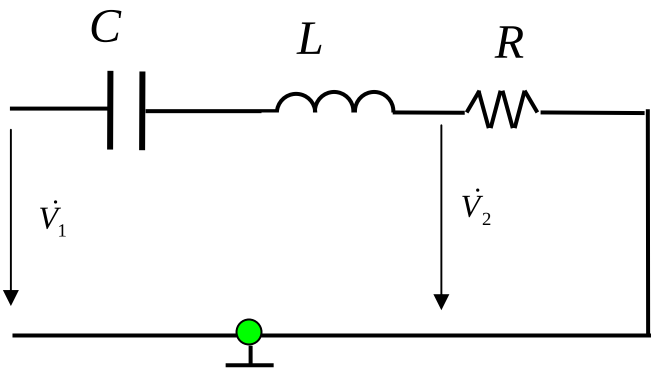


Note that coefficient a_0 of the canonic form of the transfer function is represented as:

$$a_0 = \frac{1}{\tau}$$

τ – time constant of the 1st order circuit.

Another example is RLC loop:

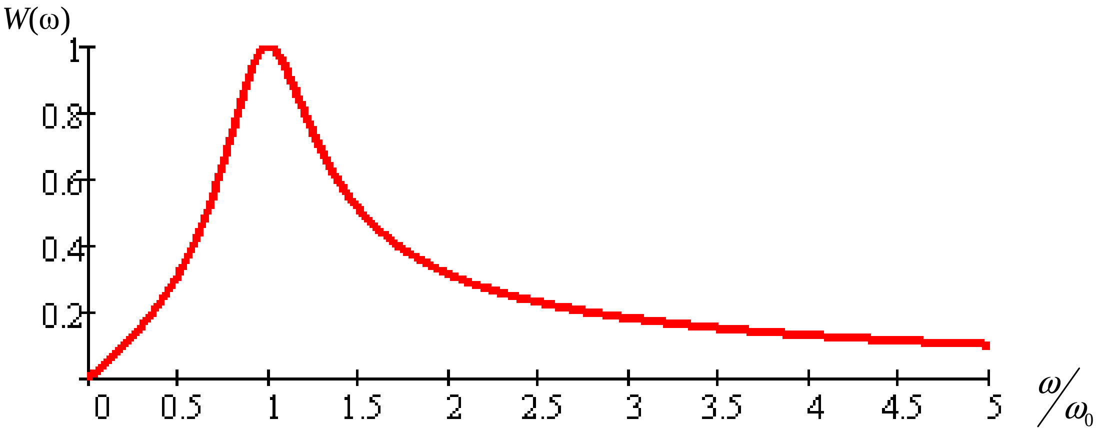


There is a transfer function:

$$W(j\omega) = \frac{R}{j\omega L + \dfrac{1}{j\omega C} + R} = \frac{j\omega \dfrac{R}{L}}{(j\omega)^2 + j\omega \dfrac{R}{L} + \dfrac{1}{LC}} = \frac{j\omega b_1}{(j\omega)^2 + j\omega a_1 + a_0} = \frac{j\omega \dfrac{\omega_0}{Q}}{(j\omega)^2 + j\omega \dfrac{\omega_0}{Q} + \omega_0^2}$$

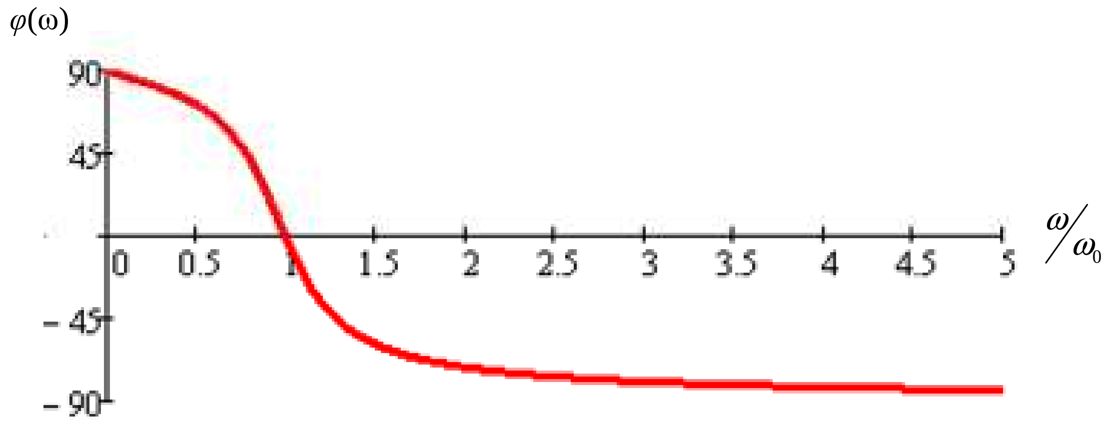
Magnitude response:

$$W(\omega) = \left| \frac{j\omega b_1}{(j\omega)^2 + j\omega a_1 + a_0} \right| = \frac{\omega b_1}{\sqrt{(a_0 - \omega^2)^2 + \omega^2 a_1{}^2}} = \frac{\omega \dfrac{\omega_0}{Q}}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \dfrac{\omega_0^2}{Q^2}}}$$

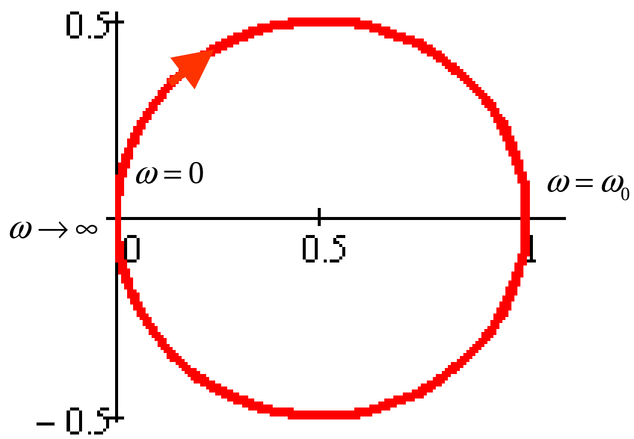


Phase response:

$$\varphi(\omega) = \arg(j\omega b_1) - \arg\left((j\omega)^2 + j\omega a_1 + a_0\right) = 90° - \arctan\left(\frac{\omega a_1}{a_0 - \omega^2}\right) = 90° - \arctan\left(\frac{\omega \dfrac{\omega_0}{Q}}{\omega_0^2 - \omega^2}\right)$$



Locus:



Note that resonance frequency and quality factor may be used for the 2nd order transfer function representation as well as coefficients of the canonic form.

Non-linear plots

Different problems require different representations of the transfer function. For example, there is a filter design problem that requires simultaneous and clear representation of amplitude response in ranges from 0,9 to 1 and from 0,00001 to 0,00001 – at the same plot and in very wide frequency range. It is necessary to use special or adaptive scales – for example, different types of the logarithmic scales.

Special units are introduced to be used in logarithmic representation:

- natural logarithm of the voltage ratio (or current ratio) is measured in Neper (Np);
- *two* decimal logarithm of the voltage ratio (or current ratio) is measured in Bell (B);
- *ten* Bells = decibel (**dB**);

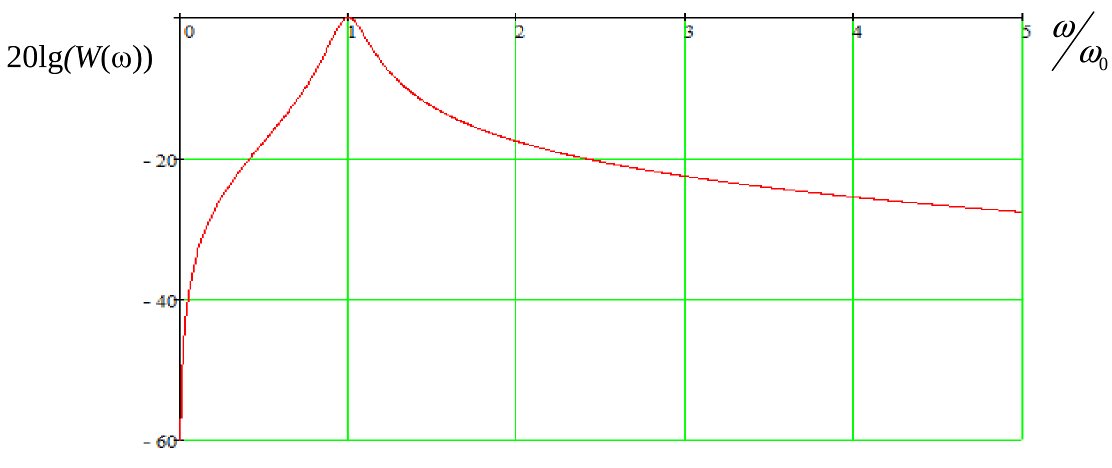
Of course, logarithm has dimensionless result, but dB is used to point out that twenty decimal logarithms are evaluated.

Simple table helps to deal with the (decimal) logarithmic scales:

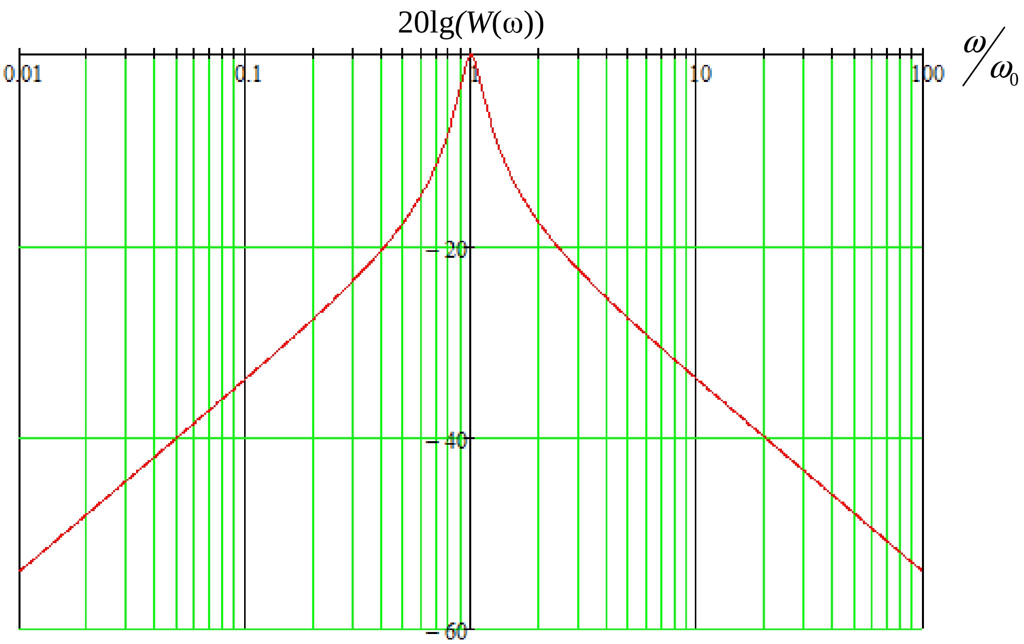
Signals ratio (argument of the logarithm)	dB
1	0
2	6 (more precise value 6,021)
$4 = 2^2$	12
square root of 2	3
$8 = 2^3$	18
cubic root of 2	2
10	20
$0,5 = 2^{-1}$	-6
0,1	-20
0,01	-40
0,001	-60
0,0001	-80
10^{-5}	-100
0	$-\infty$

Logarithmic plot are used to represent ratio of the signals, so measuring voltage it is necessary to choose *reference level* (denominator of the ratio), e.g. 1 V or 1 mV. Engineers use ordinary dB abbreviation for 1 V, 1 A, 1 W reference levels and dBm for 1 mV, 1 mA, 1 mW reference levels. Transfer function is already ratio of the voltages, so we can use logarithmic scale for amplitude response directly.

Let's redraw previous plot in the semi-logarithmic scale – it is called Bode plot (it is *semi-logarithmic* plot, because only one logarithmic scale is used):



Then, redraw this plot in the double-logarithmic scale – it is called Nyquist plot (resonance frequency is used as *reference frequency*):



Conclusion

Transfer function is a basic representation of the SISO system. It is used not only in electrical engineering but in all kinds of sciences. It will be shown later that transfer function provides full description of the dynamic system and allows to compute time-domain response of the circuit caused by arbitrary input signal. There are many different techniques of analysis and synthesis of dynamic systems (control systems, hydrodynamic systems, digital signal processing systems, thermodynamic systems and etc.).

Lecture 12.

Fourier series. Harmonic analysis. Fourier transform.

Introduction

Previous lecture describes technique of the system analysis – transfer function and frequency response. Transfer function represents *how* sinusoidal signal passes trough the SISO system (signal is associated with voltage in this case). In fact, sinusoidal signal doesn't carry information. *Absence* or *presence* of the sinusoidal voltage *does*, but it won't be sinusoidal signal – it may be considered as a signal with variable amplitude (same thing may be said about variable phase, variable frequency, etc.). This lecture describes fundamental technique of the signal analysis – Fourier (harmonic) analysis. Starting from simple Fourier series for periodical signals this lecture guides to the universal technique – Fourier transform.

Fourier series

Every *periodical* function (with the period *T*) which satisfies *Dirichlet condition* (there is *finite* number of extremum points and discontinuities on every finite interval) may be represented as **Fourier series**:

$$v(t) = V_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_S t) + b_k \sin(k\omega_S t)$$

$$\omega_S = \frac{2\pi}{T}$$

where V_0, a_k and b_k are the coefficients of the series that are computed as:

$$V_0 = \frac{1}{T}\int_o^T v(t)dt$$

$$a_k = \frac{2}{T}\int_o^T v(t)\cos(k\omega_S t)dt$$

$$b_k = \frac{2}{T}\int_o^T v(t)\sin(k\omega_S t)dt$$

Note that V_0 is just an average value of the function.

It is important that in electrical engineering and other technical sciences at almost all signals satisfy Dirichlet condition. As an example of the signal that *doesn't* satisfy this condition one can mention signal that includes $sin(x^{-1})$.

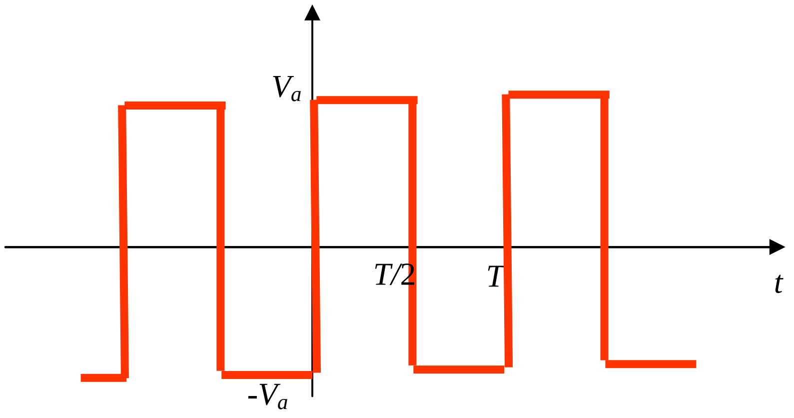
Note that $v(t)$ is used as a function (not an abstract $f(x)$, nor $i(t)$) – it just reflects the fact that the voltage is frequently associated with a signal in electrical engineering, radio engineering and electronics.

There are other possible representations of the Fourier series:

$$v(t) = V_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_S t) + b_k \sin(k\omega_S t) = V_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_S t + \varphi_k) = V_0 + \sum_{k=1}^{\infty} c_k \sin(k\omega_S t + \psi_k)$$

$$c_k = \sqrt{a_k^{\ 2} + b_k^{\ 2}} \qquad \varphi_k = \arctan\left(-\frac{b_k}{a_k}\right) \qquad \psi_k = \arctan\left(\frac{a_k}{b_k}\right)$$

As an example, let's consider periodical sequence of the square pulses (it is also called "*meander*" after the Ancient Greek amphora pattern):



Let’s find the coefficients:

$$V_0 = \frac{1}{T}\int_o^T v(t)dt = \frac{1}{T}\int_o^{\frac{T}{2}} V_a dt - \frac{1}{T}\int_{\frac{T}{2}}^{T} V_a dt = \frac{1}{T}V_a\left(\frac{T}{2}-0\right) - \frac{1}{T}V_a\left(T-\frac{T}{2}\right) = 0$$

$$a_k = \frac{2}{T}\int_o^T v(t)\cos(k\omega_S t)dt = \frac{2}{T}\int_o^{\frac{T}{2}} V_a \cos(k\omega_S t)dt - \frac{2}{T}\int_{\frac{T}{2}}^{T} V_a \cos(k\omega_S t)dt =$$

$$= \frac{2V_a}{Tk\omega_S}\left[\sin\left(k\omega_S \frac{T}{2}\right) - 0 - \sin(k\omega_S T) + \sin\left(k\omega_S \frac{T}{2}\right)\right] = 0$$

$$b_k = \frac{2}{T}\int_o^T v(t)\sin(k\omega_S t)dt = \frac{2}{T}\int_o^{\frac{T}{2}} V_a \sin(k\omega_S t)dt - \frac{2}{T}\int_{\frac{T}{2}}^{T} V_a \sin(k\omega_S t)dt =$$

$$= \frac{-2V_a}{Tk\omega_S}\left[\cos\left(k\omega_S \frac{T}{2}\right) - \cos(0) - \cos(k\omega_S T) + \cos\left(k\omega_S \frac{T}{2}\right)\right] = \begin{cases} \frac{8V_a}{Tk\omega_S} = \frac{4V_a}{k\pi} & k = 1,3,5,7,... \\ 0 \quad k = 2,4,6,... \end{cases}$$

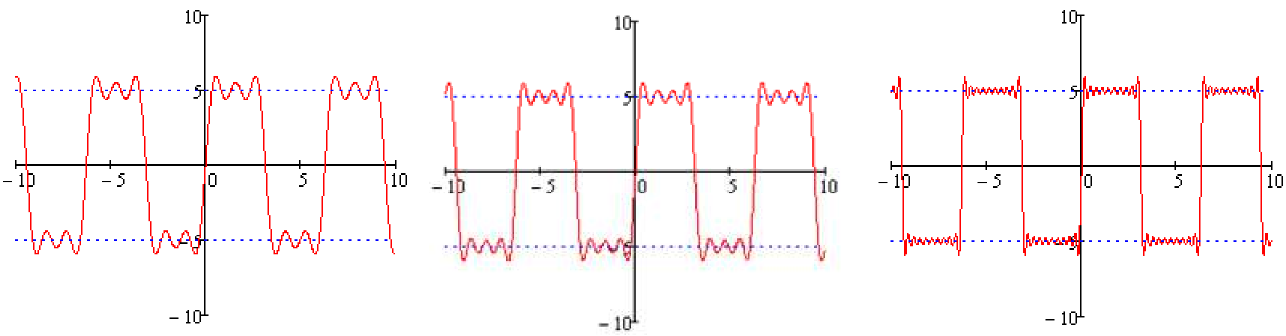
Note that there are non-zero coefficients b_k (and just for odd k’s), so one can rewrite Fourier series for meander in the following form:

$$v(t) = \frac{4V_a}{\pi}\sum_{k=1,3,5,7...}^{\infty}\frac{\sin(k\omega_S t)}{k}$$

This is, in fact, simple representation of the simple signal, but, evidently, the *reconstruction* of the original signal requires *infinite* number of the series coefficients. Thus, restricting series at finite number n of coefficients we find approximate formula:

$$v(t) \approx \frac{4V_a}{\pi}\sum_{k=1,3,5,7...}^{n}\frac{\sin(k\omega_S t)}{k}$$

Following figures show how n influences on the waveform of the signal:



V_a=5, n=5 n=7 n=19

Each sinusoidal component of the series is called **harmonic** component (as sinusoidal function is also called harmonic function). Thus n is called **number of harmonics**. This is why we call this signal representation “**harmonic analysis**”.

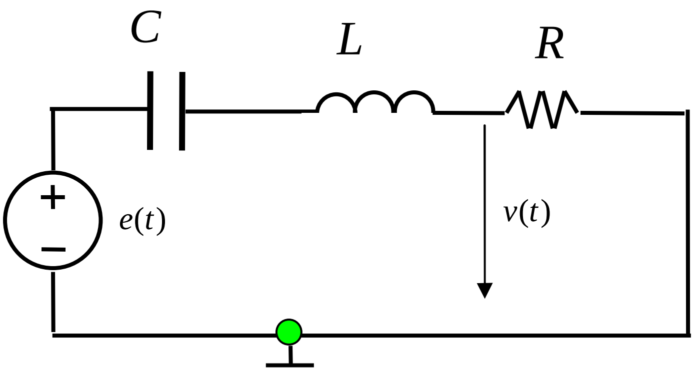
Reasonable number of harmonics needed for analysis may be determined by different ways: convergence of the result (waveform, total power, etc.), physical restriction of *higher harmonics* in the circuit or something else…

Harmonic analysis and AC analysis

AC analysis with effective complex representation was introduced for sinusoidal waveform and harmonic analysis expands application area of the AC analysis to all periodical signals. Generally speaking AC analysis covers linear circuits only, but harmonic analysis may be successfully applied to non-linear circuits (one can mention harmonic balance method – powerful tool of the non-linear circuit analysis).

Superposition principle allows to use complex representation (and all features of the AC analysis – complex impedance, admittance, complex power balance, etc.) of voltages and currents for *each harmonic* component of the Fourier series *separately*.

As an example, consider RLC circuit and the meander voltage (number of harmonics is 5):



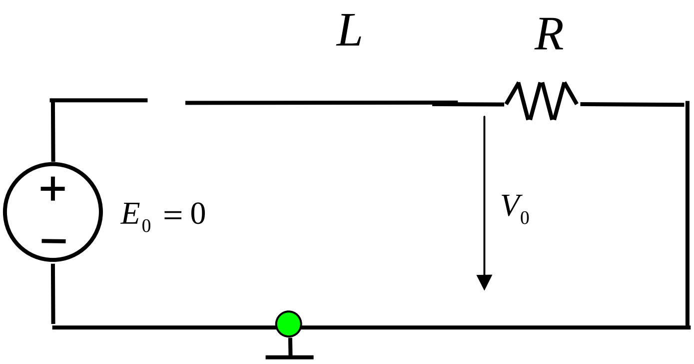
$$e(t) \approx \frac{4E_a}{\pi} \sum_{k=1,3,5}^{5} \frac{\sin(k\omega_S t)}{k}$$

$\omega_S = 10^3 \ s^{-1}, \quad L = 1\,H, \quad C = 0{,}25\mu F$

$R = 100\ \Omega$

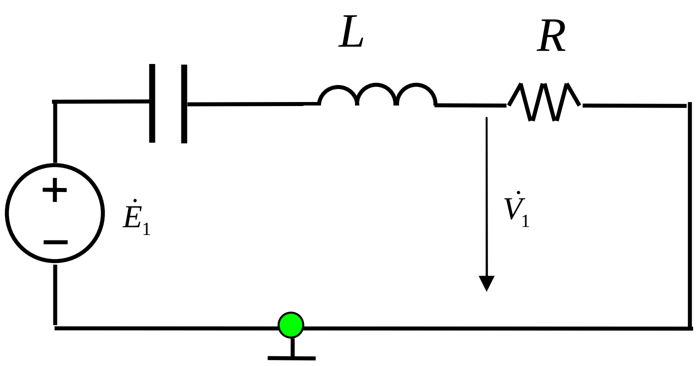
$E_a = 5V$

DC analysis for V_0:



$V_0 = 0$

AC analysis for the 1st harmonic:



$$\dot{E}_1 = \frac{20}{\pi}\angle 0°$$

$$\frac{1}{\omega_S C} = 4000 \quad \Omega, \quad \omega_S L = 1000 \quad \Omega$$

$$\dot{V}_1 = \frac{20}{\pi}\angle 0° \frac{R}{R + j\omega_S L - j\dfrac{1}{\omega_S C}} \cong 2\angle 71°$$

AC analysis for the 3rd harmonic:

$$\dot{E}_3 = \frac{20}{3\pi}\angle 0°$$

$$\frac{1}{3\omega_S C} = 1333 \quad \Omega, \quad 3\omega_S L = 3000 \quad \Omega$$

$$\dot{V}_3 = \frac{20}{3\pi}\angle 0° \frac{R}{R + j3\omega_S L - j\dfrac{1}{3\omega_S C}} \cong 1{,}1\angle -59°$$

AC analysis for the 5th harmonic:

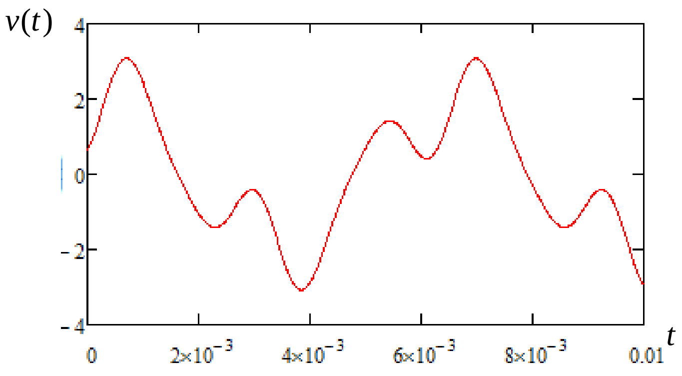
$$\dot{E}_5 = \frac{20}{5\pi}\angle 0°$$

$$\frac{1}{5\omega_S C} = 800 \quad \Omega, \quad 5\omega_S L = 5000 \quad \Omega$$

$$\dot{V}_3 = \frac{20}{5\pi}\angle 0° \frac{R}{R + j5\omega_S L - j\dfrac{1}{5\omega_S C}} \cong 0{,}3\angle -77°$$

Total result:

$$v(t) = 2\sin(\omega_S t + 71°) + 1{,}1\sin(3\omega_S t - 59°) + 0{,}3\sin(5\omega_S t - 77°)$$



Complex representation of the Fourier series

As the complex representation is very convenient tool of the AC analysis, it may be used for harmonic analysis:

$$v(t) = V_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \dot{V}_k e^{jk\omega_S t}$$

Note that complex representation extends range of the series from [0..∞) to (-∞..∞) and complex coefficients with negative indices are conjunctions of the coefficient with positive indices:

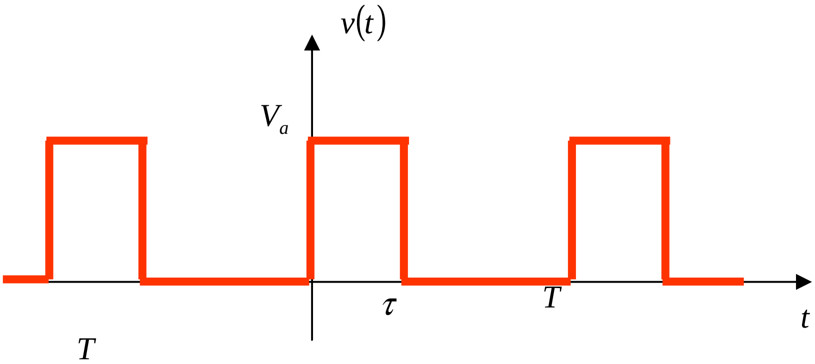
$$\dot{V}_k = \left(\dot{V}_{-k}\right)^*$$

Complex coefficients may be found as:

$$\dot{V}_k = \frac{2}{T}\int_0^T v(t) e^{-jk\omega_S t} dt$$

Sequence of the complex coefficients magnitudes forms **discrete amplitude spectrum** of the signal. It is discrete because magnitudes correspond to discrete indices *i*, not the continuous function of frequency. Similarly, sequence of the complex coefficients arguments forms **discrete phase spectrum** of the signal.

As an example, consider sequence of rectangular pulses. **Off-duty ratio** *m* is defined as ratio of the period to pulse width (e.g. meander has *m* = 2):

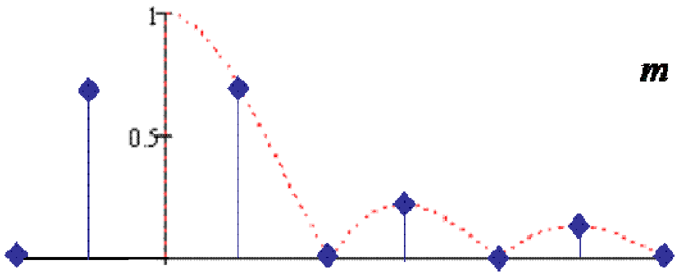


$$m = \frac{T}{\tau}$$

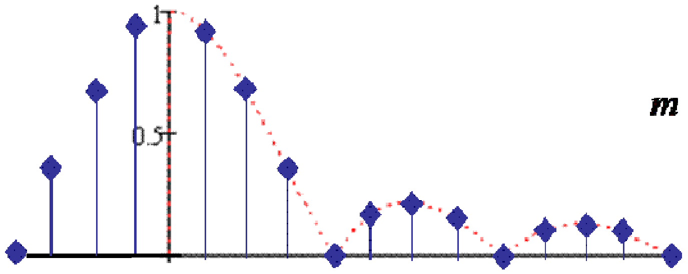
This is common formula that gives discrete magnitude and phase spectra:

$$\dot{V}_k = \frac{2}{T}\int_0^T v(t)e^{-jk\omega_S t}dt = \frac{2}{T}\int_0^\tau V_a e^{-jk\omega_S t}dt = -\frac{2V_a}{Tk\omega_S}\left(e^{-jk\omega_S\tau} - 1\right) = -\frac{V_a}{k\pi}\left(e^{-jk\omega_S\tau} - 1\right)$$

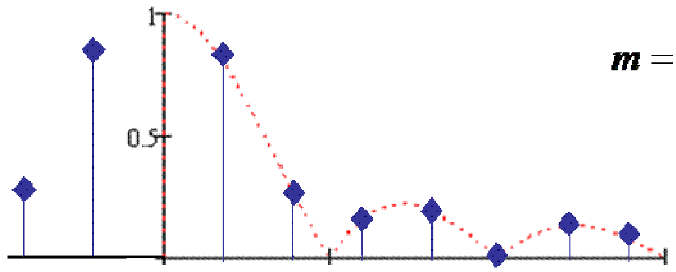
Following pictures show amplitude spectra of the signals with different off-duty ratio:



$$m = \frac{T}{\tau} = 2$$



$$m = \frac{T}{\tau} = 4$$



$$m = \frac{T}{\tau} = 2{,}5$$

Fourier transform

Fourier series is applicable to periodical signals. When it is non-periodical signal is given (or it isn't exactly known if the signal is periodical), another, but very similar technique may be used. Non-periodical signal is represented as *periodical with infinite period*. It allows to use ordinary complex Fourier series with certain modification:

$$v(t) \underset{\substack{T\to\infty \\ \omega_S\to 0}}{=} V_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\dot{V}_k e^{jk\omega_S t} \Rightarrow v(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\dot{V}(\omega)e^{j\omega t}d\omega$$

Thus, sums are replaced by the integrals and discrete series (discrete spectrum) is replaced by the continuous function of the frequency (**continuous spectrum**):

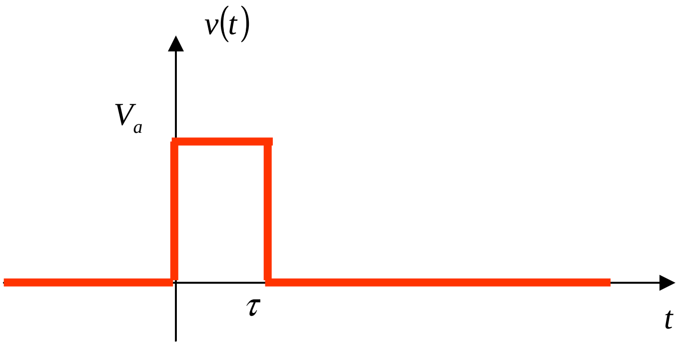
$$\dot{V}(\omega) = \int_{-\infty}^{+\infty} v(t)e^{-j\omega t}dt$$

- this formula is called **forward Fourier transform**;

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\dot{V}(\omega)e^{j\omega t}d\omega$$

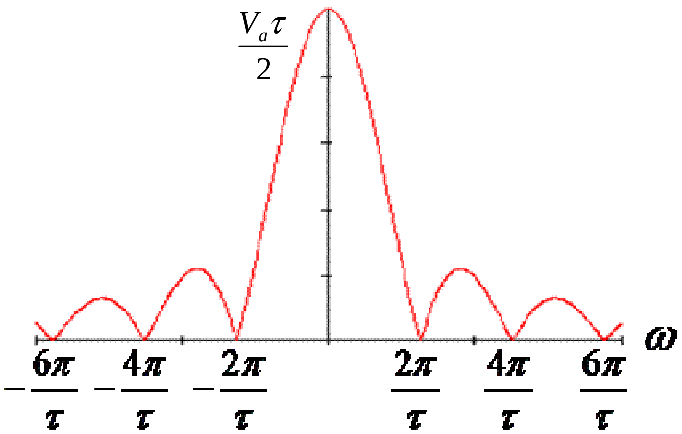
- and this expression is called **reverse Fourier transform**.

As an example, let's consider continuous spectrum of the single rectangular pulse:



$$\dot{V}(\omega) = \int_{-\infty}^{+\infty}v(t)e^{-j\omega t}dt = \int_{0}^{\tau}V_a e^{-j\omega t}dt = -\frac{V_a}{j\omega}\left(e^{-j\omega\tau} - 1\right)$$

Following figure shows continuous amplitude spectrum of the single pulse:



AC analysis doesn't help to obtain circuit response caused by the non-periodic signal. Transfer function (that can be considered as expansion of the AC analysis – in fact, we didn't study something new about complex representation or impedances or something else!) allows to compute such response. It is just necessary to multiply continuous spectrum of the signal by the transfer function:

$$\dot{V}_{OUT}(j\omega) = W(j\omega)\dot{V}_{IN}(j\omega)$$

So, the algorithm seems to be simple:

1) find spectrum of the input signal through forward Fourier transform;
2) spectrum of the output signal is a product of spectrum of the input signal and transfer function of the system;
3) reconstruct output signal from the spectrum by using reverse Fourier transform.

Filters, ideal filter, linear phase filter

Circuit, especially two-port system which is designed to *operate on* the signal *spectrum* is called **filter**. It is necessary to underline word "designed" – at almost every circuit changes spectrum of the input signal, but certain spectrum manipulation is the main and most important purpose of the filter. So that, filter is just an ordinary circuit (and ordinary analysis methods are applicable) which is designed to have specific transfer function.

Transfer function of the filter has at least two *bands* (band is specific frequency range): pass-band and stop-band. Filter transmits signals with the spectrum located within the pass-band and restricts signals which spectrum is located inside the stop-band. These properties correspond to specific requirements.

First of all, let's describe signal passage through the filter – requirements for the pass-band. It is necessary to conserve the shape (or waveform) of the signal within the pass-band. As the spectrum of the output signal is connected with the spectrum of the input signal and transfer function, one can formulate a condition of the signal passage:

$$v_{IN}(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\dot{V}_{IN}(j\omega)e^{j\omega t}d\omega,\quad v_{OUT}(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\dot{V}_{OUT}(j\omega)e^{j\omega t}d\omega$$

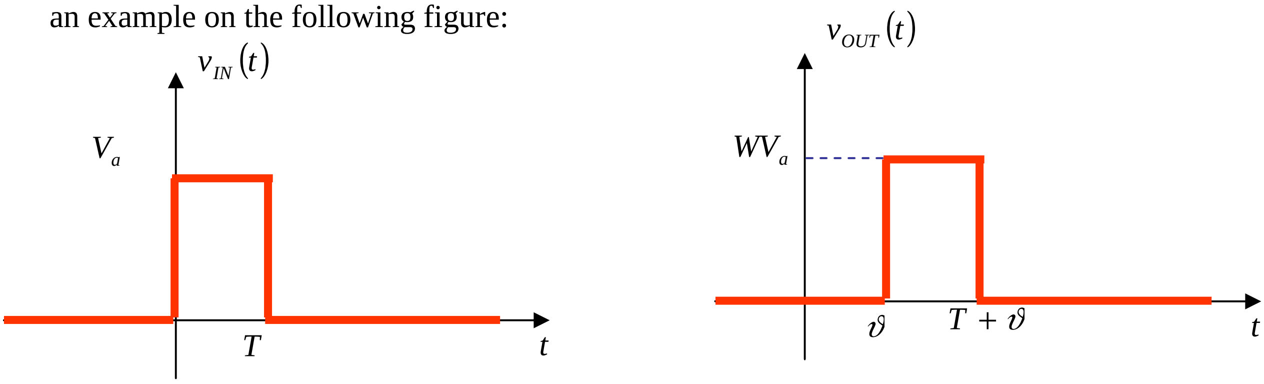
$$\dot{V}_{OUT}(j\omega)=W(j\omega)\dot{V}_{IN}(j\omega)\quad\rightarrow\quad v_{OUT}(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}W(j\omega)\dot{V}_{IN}(j\omega)e^{j\omega t}d\omega$$

$$v_{OUT}(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}W(\omega)V_{IN}(\omega)e^{j\left[\omega t+\arg(W(j\omega))+\arg\left(\dot{V}_{IN}(j\omega)\right)\right]}d\omega$$

$$W(j\omega)=We^{-j\omega\vartheta}$$

$$v_{OUT}(t)=\frac{W}{2\pi}\int_{-\infty}^{+\infty}V_{IN}(\omega)e^{j\left[\omega t-\omega\vartheta+\arg\left(\dot{V}_{IN}(j\omega)\right)\right]}d\omega=\frac{W}{2\pi}\int_{-\infty}^{+\infty}V_{IN}(\omega)e^{j\omega(t-\vartheta)+\arg\left(\dot{V}_{IN}(j\omega)\right)}d\omega=Wv_{IN}(t-\vartheta)$$

Filter should have *constant amplitude* response and *linear phase* response inside the pass-band. Output signal will have same waveform as the input one and it will be delayed by the time ϑ. See an example on the following figure:

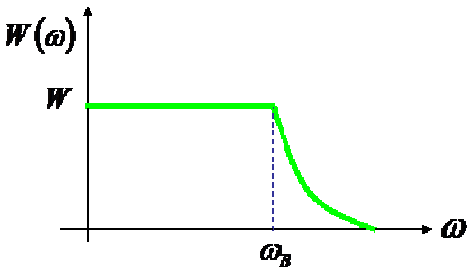


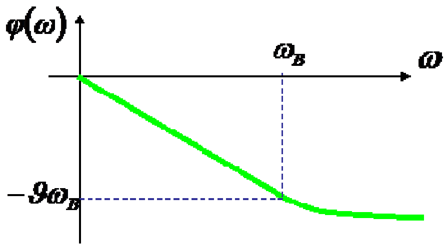
All spectral components of the input signal should be eliminated inside the stop-band – it is necessary to set the transfer function to zero in the stop-band.

Filter which has such transfer function is called **ideal filter**. As we could see there are no constant frequency responses obtained – just relatively sophisticated functions of frequency. Moreover, phase response of the circuit is a combination of the arctangent functions of polynomial ratio; it may be just *approximated* by linear function in the certain frequency range (one can refer to Taylor series – restricting all coefficient of order higher than 1, it is possible to approximate arctangent by the linear function at the selected point). Purpose of the filter design is to arrange elements of the electrical circuit and to find parameters of these elements achieving the best fit of the ideal characteristic.

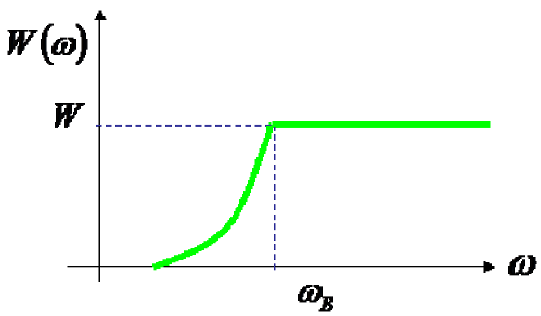
There are four basic types of the electrical filters – they arranged with accordance to the placement of the pass-band and stop-band:

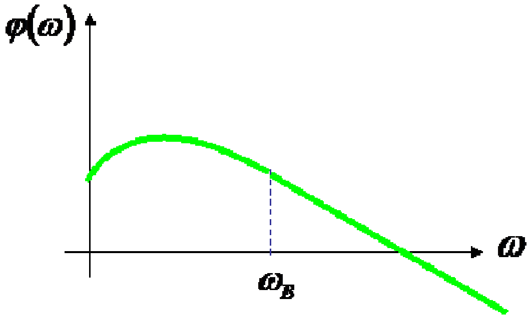
1) low-pass filter



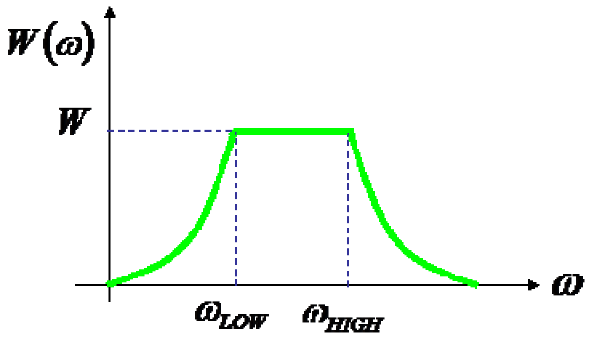


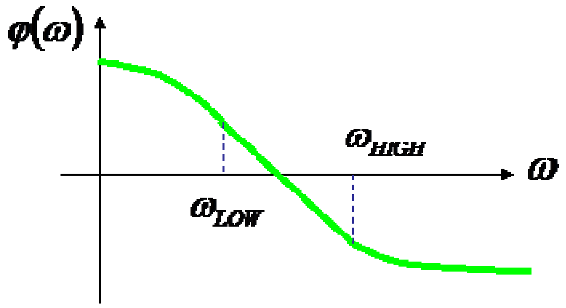
2) high-pass filter



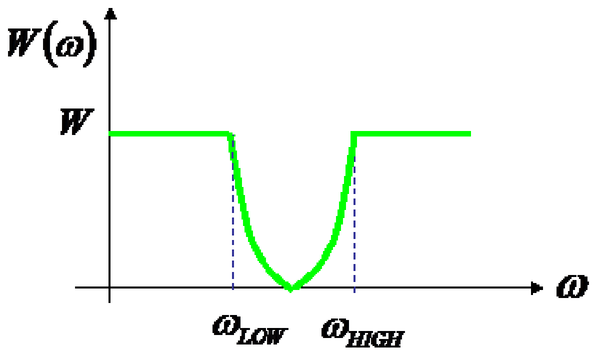


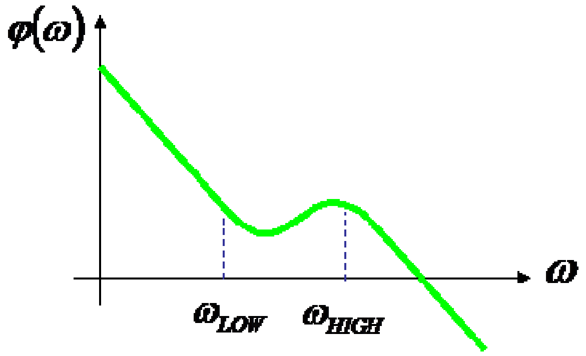
3) band-pass filter



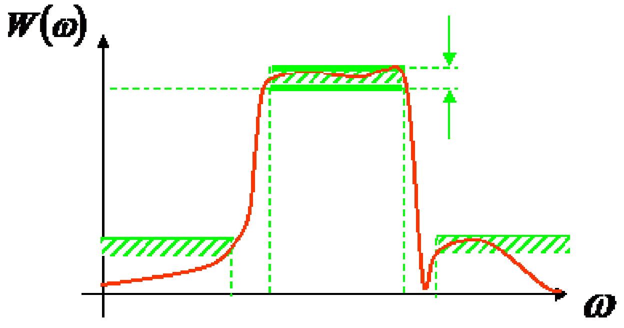


4) band-stop filter





Following figure shows real filter frequency response plotted over the ideal filter mask:



Stochastic signals

To transmit information it is necessary to use variable signal – it may change waveform or different parameters according to the given information (e.g. sequence of “ones” and “zeros” in binary signal). Claude Shannon introduced quantity of information (in binary signal) as a value that depends on probability of “1” or “0”. If the signal is known *a priory* it doesn’t carry any quantity of information. For example, “1” corresponds to pulse of known duration and “0” corresponds to zero signal. We know spectrum of the single pulse or spectrum of the periodic pulses sequence, but it is unknown a priory (before receiving of the message) which sequence corresponds to the signal. This type of signals is called stochastic signals as just a probability of the signal’s parameters may be evaluated. This leads to stochastic spectrum – it may be considered as a probability distribution or *average* spectrum during specific time period. Of course averaging interval has great influence on the spectrum and it is relatively difficult problem to determine right time period. Multiply analysis of the stochastic signal is usually required.

Conclusion

Fourier transform and harmonic analysis are powerful tools of electrical engineering: in power engineering – harmonic analysis is the base of the electrical energy quality development; in electronics – harmonic analysis is widely used to describe characteristics of analog ICs (amplifiers, transmitters, …), to design of non-linear generators; in radio engineering – it is just a fundamental (better to say – the only) technique of RF (radio frequency) analysis; in automatic control and digital signal processing, information theory and acoustics – see radio engineering! In fact, Fourier transform (harmonic analysis) is a kind of a language of the modern engineering.

Lecture 13.

Transient analysis. First order ODE. Forced and natural solutions.

Introduction

This lecture starts a new part of our course – time-domain (transient) analysis. Time domain analysis is used to describe dynamics of the circuit. Dynamic system is something that is described by differential equations. One can conclude that this lecture starts more difficult part of the course, because it is evident that solution of the algebraic equations seems to be easier than solution of the differential ones!

Reasons of the dynamic processes

Why do we use time-domain analysis? We’ve already studied DC and AC analysis, Fourier transform – so that, we can compute responses for all kinds of input signals (and without any differential equations!). It may be said that time-domain analysis is a basic technique that describes processes in the circuit caused by any changes in the circuit – sources may be changed, or parameters of elements may vary, or topology of the circuit may be changed.

It is important thing, that circuit cannot change its state instantaneously. The reason is, in fact, the conservation of energy law. If there is a charged capacitor in a circuit, energy that is stored in the electric field is:

$$W_E(t) = C\frac{v^2(t)}{2} \quad \rightarrow \quad W_E(0-) = C\frac{v^2(0-)}{2}$$

at the moment t = 0-. "0-" means left-side limit:

$$W_E(0-) = \frac{C}{2}\lim_{t\to 0 left} v^2(t)$$

At moment t = 0 something happened in the circuit, something that force it to change the state, to change all the voltages and currents. What is the energy of the capacitor at moment t = 0+? "0+" means right-side limit:

$$W_E(0+) = \lim_{t\to 0\, right} W(t)$$

To change the energy it is necessary to apply a power:

$$\Delta W_E = W_E(0+) - W_E(0-) = P\Delta t, \quad \Delta t \rightarrow 0$$

Power must be finite function, so energy must be continuous function of time. If there is constant value of the capacitance, one can claim that voltage on the capacitance is continuous function of time:

$$W_E(0+) = W_E(0-) \quad \rightarrow \quad v(0+) = v(0-)$$

Same thing may be said about inductance; the only difference is that the magnetic field energy corresponds to current:

$$W_M(t) = L\frac{i^2(t)}{2} \quad \rightarrow \quad i(0+) = i(0-)$$

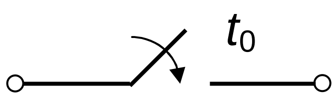
One may conclude:

- voltage on the capacitance cannot be changed instantaneously;
- current in the inductance cannot be changed instantaneously.

These phrases are known as **commutation laws**.

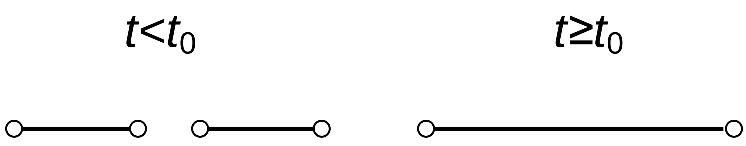
Switcher closure

To determine different *instantaneous* changes in the circuit uniformly, special element is introduced – ideal **switcher**.



Arrow points the movement direction – from previous position to the next. There is a time moment of the switcher closure specified nearby the switcher. For example, if the switcher closure time $t = 0$ is specified, $t = 0-$ is the moment *before* closure, $t = 0$ or $t = 0+$ is the moment *after* closure.

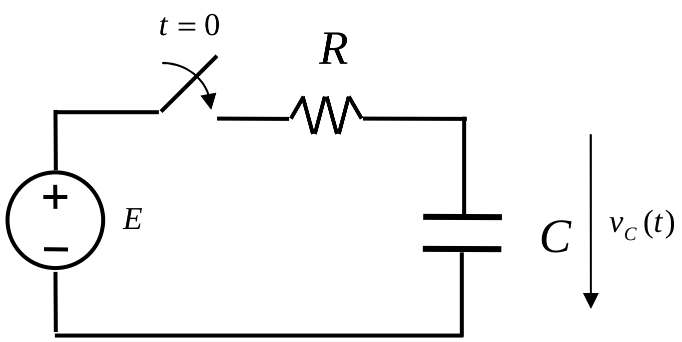
Ideal switcher provides perfect connection (with zero resistance) being "turned on" and perfect insulation (with zero conductance) being "turned off". So it may be represented by the short circuit or open circuit before and after the closure. Note that closure is instantaneous process, it lasts zero time.



Differential equation of the circuit (1st order)

There is no common method of the differential equation formulation. Each circuit requires individual analysis and, in fact, this problem seems to be the most difficult part of the overall time-domain analysis. One can combine KVL and KCL with constitutive relation of the elements to obtain a differential equation of the given circuit. It may take several attempts, it may be necessary to differentiate equations and to make sophisticated substitutions. Note that the result is a differential equation which contains only one unknown function (current or voltage) – e.g. constitutive relation of the inductance *is not* a differential equation (as it contains two unknown functions), but just a differential expression, formula, dependence.

Let's consider simple example – constant e.m.f. source is being connected to the RC loop. Capacitance was uncharged before switcher closure (so that, voltage on it was equal to zero):



There is the only loop in the circuit after the switcher closure and there are no nodes. One can use KVL once:

$$-E + iR + v_C = 0$$

This expression is not a differential equation – it consists of known variable E and pair of unknown functions i and v_C. To obtain a differential equation it is necessary to use constitutive relation for the capacitance:

$$i = C\frac{dv_C}{dt}$$

$$RC\frac{dv_C}{dt} + v_C = E$$

There is a *linear non-homogeneous ordinary differential equation (ODE) of the first order with the constant coefficients* for unknown function v_C.

Several important notes:

- this equation is *linear* because it doesn't contain expressions like v_C^2 or $ln(v_C)$ (but it may contain t^2 or $ln(t)$ in the right hand);
- this equation is *non-homogeneous* because the right hand expression isn't equal to zero (otherwise, it would be homogeneous equation);
- this equation is called *ordinary* differential equation because it contains derivatives with respect to time only (if there were derivatives with respect to x and y, it would be partial differential equation);
- *order* of an equation is determined by the highest order of the derivatives;
- *constant coefficients* (not variable coefficients) are RC and 1;
- the process of the solution of the ODE is also called *integration*;
- solution of this ODE is function $v_C(t)$.

It is important to specify exact type of the differential equation because different types correspond to completely different solution methods.

Solution of the linear non-homogeneous ODE with constant coefficients consists of two parts: general solution of homogeneous equation and particular solution of non-homogeneous equation (shorter terms are "*general solution*" and "*particular solution*"):

$$v_C(t) = v_{CG}(t) + v_{CP}(t)$$

General solution

General solution corresponds to homogeneous equation, so it is necessary to put right-hand expression to zero:

$$RC\frac{dv_C}{dt} + v_C = 0$$

A characteristic equation is obtained from the homogeneous ODE by the simple substitution:

$$\frac{d}{dt} \to \lambda$$

So, in this particular case, characteristic equation is:

$$RC\lambda + 1 = 0$$

As it is first order algebraic equation, it has the only root:

$$\lambda = \frac{-1}{RC}$$

General solution of the homogeneous ODE is:

$$v_{CG}(t) = Ae^{\lambda t}$$

where *A* – integration constant (this constant will be evaluated later).

As the first order ODE always corresponds to the only root of the characteristic equation, this root is usually substituted by simple expression:

$$\lambda = -\frac{1}{\tau}$$

where τ = *RC* is the time constant of the circuit. Then, general solution may be represented as:

$$v_{CG}(t) = Ae^{-t/\tau}$$

Note that root of the characteristic equation has negative value, so exponential function of the negative argument declines and approaches zero at infinite time.

Particular solution

In fact, there are no formal analytical particular solutions available for the arbitrary right hand expression. The simplest algorithm is just to guess, which function satisfies ODE with given right hand expression. Of course, right hand expression always helps to guess. In our example, there is a constant in the right hand, so one can make an attempt to find particular solution as a constant:

Assume $v_{CP} = V = const$; substitute v_C with V in the ODE:

$$RC\frac{dv_{CP}}{dt} + v_{CP} = E$$

$$RC\frac{dV}{dt} + V = E \quad \rightarrow \quad V = E$$

So that, particular solution is found: $v_{CP} = E$.

Initial condition

There is an unknown integration constant *A*:

$$v_C(t) = E + Ae^{-t/\tau}$$

To find the value of the integration constant it is necessary to use *additional* information – initial condition (evaluation of the integration constant is “electrical” problem, not formal “mathematical” problem). One can use commutation law to find initial voltage on the capacitance:

$$v_C(0+) = v_C(0) = v_C(0-) = 0$$

So if there was zero voltage on the capacitance before switcher closure, it stays zero at the t = 0+ after closure.

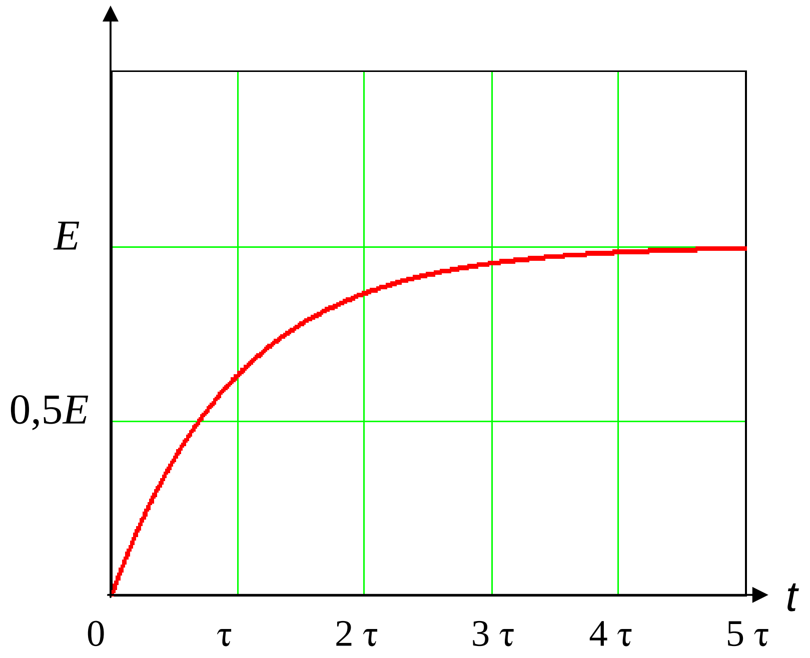
At t = 0 our solution gives us:

$$v_C(0) = E + A$$

Thus, A=-E. Final formula for the capacitance voltage is:

$$v_C(t) = E\left(1 - 1e^{-t/\tau}\right)$$

Following figure shows the curve of the voltage:



An example (capacitance current)

Let's consider differences of the solution for the current in the same circuit. To form the ODE for the current it is necessary (in this particular case) to differentiate previous equation:

$$RC\frac{d^2v_C}{dt^2} + \frac{dv_C}{dt} = \frac{dE}{dt}$$

As constitutive relation gives

$$i = C\frac{dv_C}{dt}$$

One can substitute first derivatives of the capacitance voltage by the current, divided by the capacitance C:

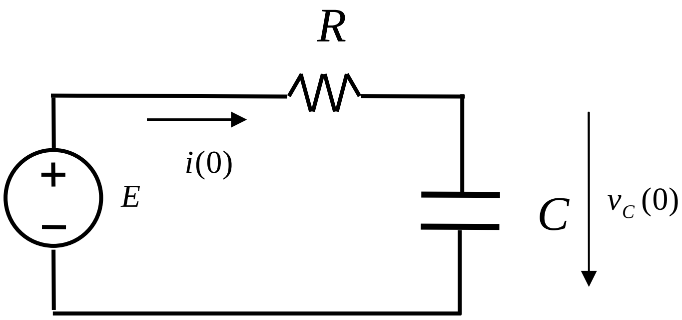
$$R\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

There is linear homogeneous ODE with constant coefficients. As it is homogeneous, there is no need to find particular solution. It is evident, that there is same characteristic equation as in a previous case. It is important property of the characteristic equation – this equation doesn't depend on the choice of the unknown function. In fact, roots of the characteristic equation are *fundamental constants* of the circuit.

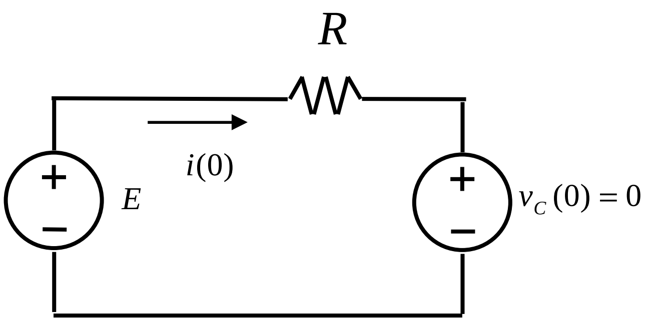
Thus, solution for the current is:

$$i(t) = Be^{\lambda t}$$

Integration constant B should be found through the initial condition $i(0)$. It was easy to find $v_C(0)$, but there is no commutation law for the current in the capacitance or in the resistance. It is necessary to consider circuit at $t = 0$ (after the switcher closure):



As the voltage on the capacitance at zero time is known, one can use *compensation principle* and substitute capacitance with the e.m.f. source of the known voltage:



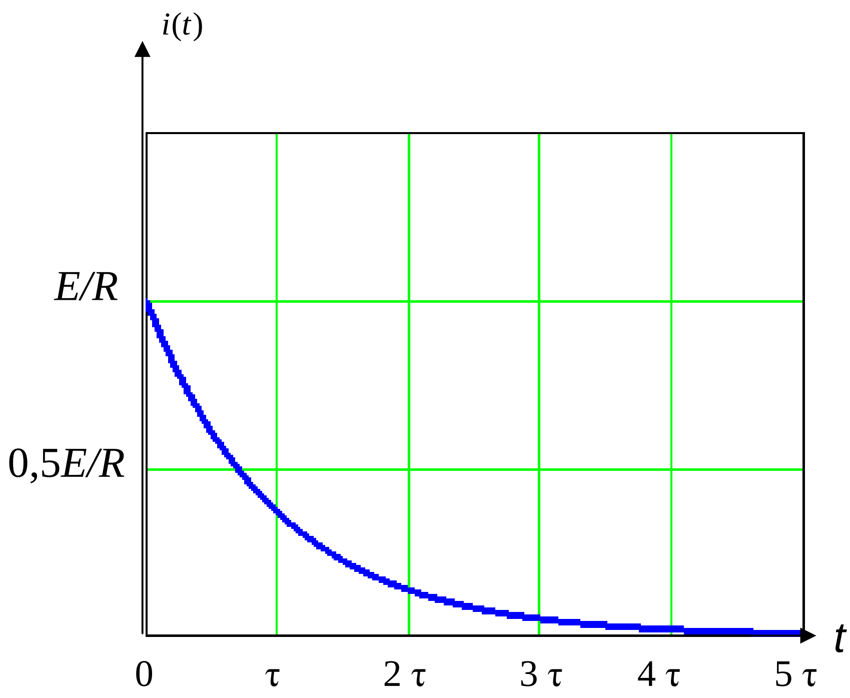
KVL leads us to the simple expression:

$$-E + i(0)R + v_C(0) = 0$$

Thus $i(0)=E/R=B$ and final formula is:

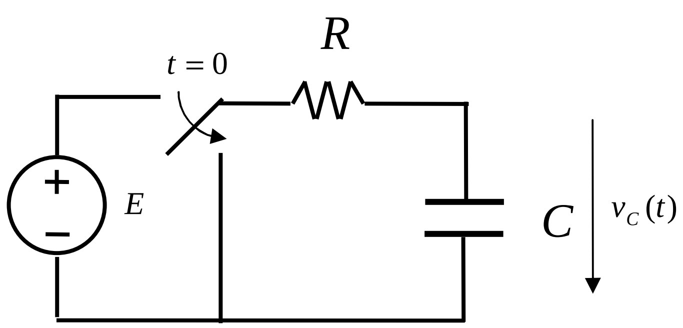
$$i(t) = \frac{E}{R} e^{-t/\tau}$$

Following figure shows the plot of the current in the circuit:

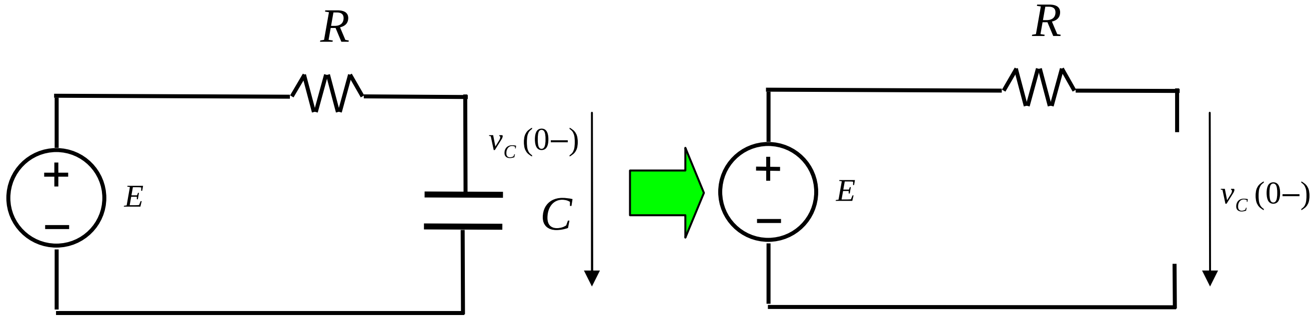


Another example (capacitance discharge)

Let's consider "reverse" switcher closure:

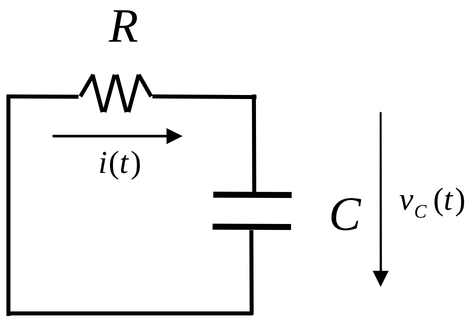


First of all, it is necessary to find the voltage on the capacitance *before* the switcher closure:



Note that circuit was in the *steady-state* mode before the closure. Therefore, capacitance may be substituted by the *open circuit*. Voltage on the capacitance before the closure is equal to the e.m.f. E.

After the switcher closure:



One can use KVL to form an equation:

$$i(t)R + v_C(t) = 0$$

Constitutive relation allows to obtain linear *homogeneous* ODE of the 1st order with the constant coefficients for v_C:

$$RC\frac{dv_C}{dt} + v_C = 0$$

or for i:

$$RC\frac{di}{dt} + i = 0$$

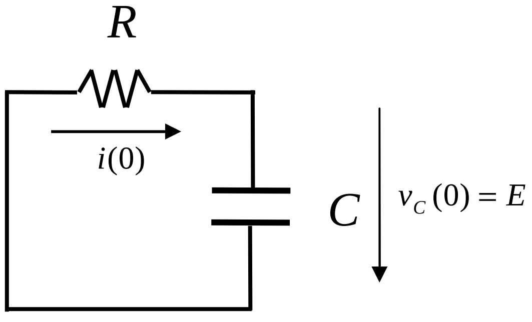
Characteristic equation:

$$RC\lambda + 1 = 0$$

has the only root:

$$\lambda = -\frac{1}{RC}$$

Initial condition for the capacitor was found above. Initial condition for the current may be obtained in the scheme of the circuit at $t = 0$:

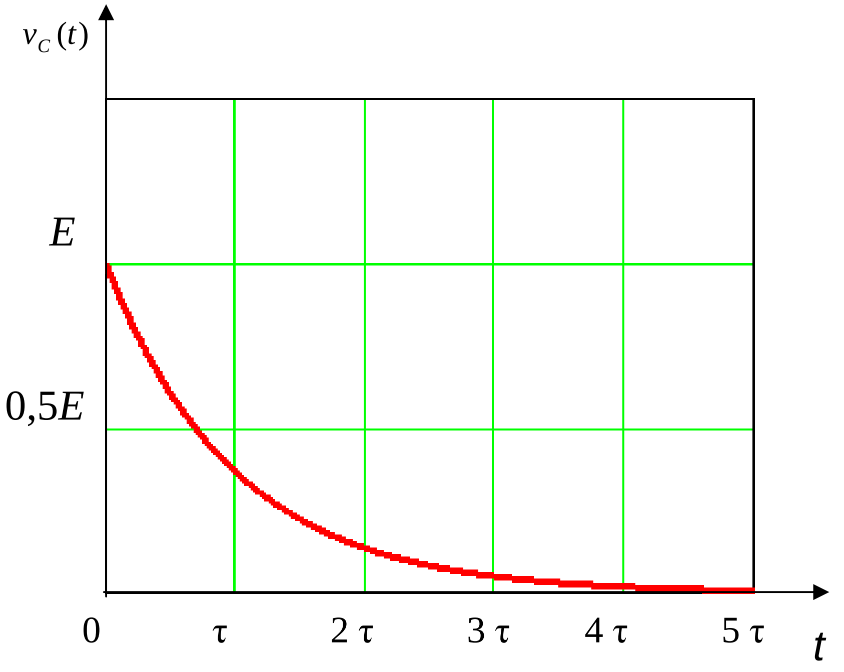


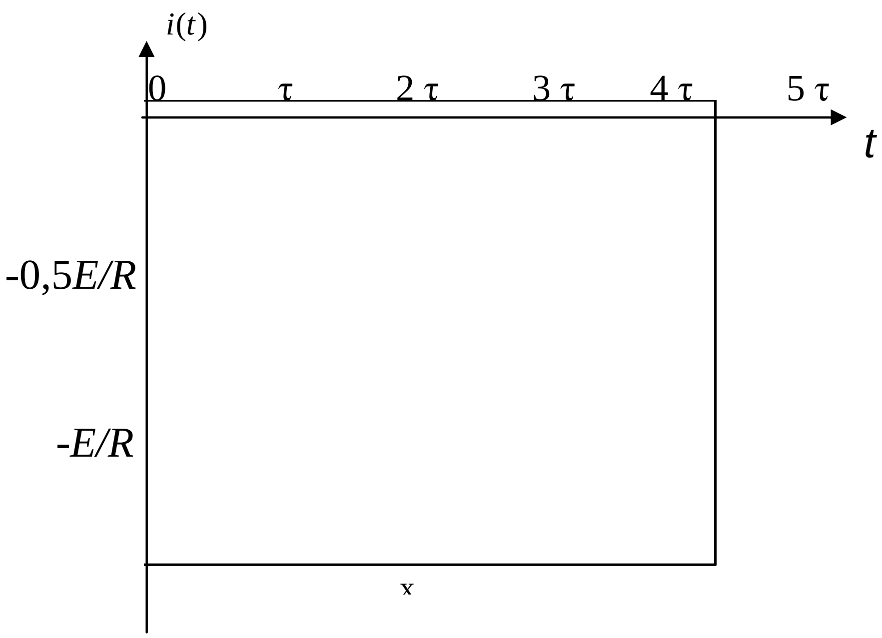
Note that capacitance is substituted by the e.m.f. source $v_C(0)$. Current is equal to $i(0) = -v_C(0)/R = -E/R$. Final expressions are:

$$v_C(t) = Ee^{\lambda t}$$

$$i(t) = -\frac{E}{R}e^{\lambda t}$$

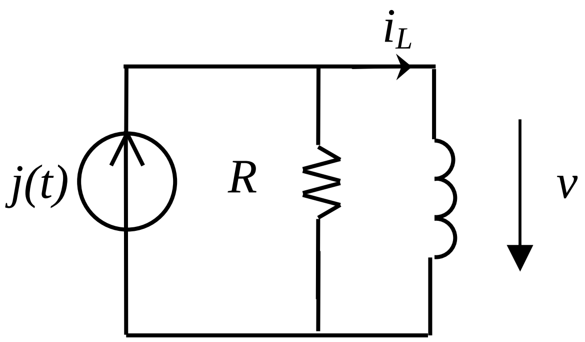
Following figure shows the plots of the found functions:

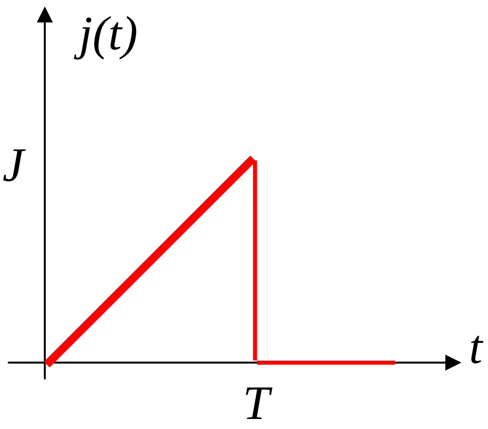




The last example (current in the inductance)

Let's consider parallel *RL* circuit which is connected to the *ramp* current source, and then, disconnected:





First switcher closure:

1) Before the closure current in the inductance was equal to zero $i_L(0\text{-})=0$.
2) After the closure ODE may be obtained by using KCL, constitutive relation and Ohm's law:

$$j(t) = i_L(t) + v(t)g \quad \rightarrow \quad \left[v = L\frac{di_L}{dt} \right] \quad \rightarrow \quad Lg\frac{di_L}{dt} + i_L = j$$

3) Characteristic equation corresponds to the homogeneous ODE:

$$Lg\frac{di_L}{dt} + i_L = 0 \quad \rightarrow \quad gL\lambda + 1 = 0 \quad \rightarrow \quad \lambda = -\frac{1}{Lg} = -\frac{R}{L} \quad \tau = \frac{L}{R}$$

4) Particular solution may be found as a linear function (as the right hand expression is a linear function of time):

$$i_{Lp}(t) = at + b$$

$$Lg\frac{di_{Lp}}{dt} + i_{Lp} = \frac{J}{T}t \quad \rightarrow \quad Lga + at + b = \frac{J}{T}t$$

$$\begin{cases} at = \frac{J}{T}t \\ Lga + b = 0 \end{cases} \rightarrow \begin{cases} a = \frac{J}{T} \\ b = -Lg\frac{J}{T} \end{cases}$$

5) Initial condition is found in accordance with commutation law:

$$i_L(0) = i_L(0-) = 0$$

6) Final formula is:

$$i_L(t) = at + b + Ae^{\lambda t} = \frac{J}{T}t - Lg\frac{J}{T} + Lg\frac{J}{T}e^{-\frac{t}{Lg}}$$

Second switcher closure:

1) Before the closure (at the moment t = T-) inductance current may be found by the previously obtained formula:

$$i_L(T-) = \frac{J}{T}T - Lg\frac{J}{T} + Lg\frac{J}{T}e^{-\frac{T}{Lg}}$$

2) After the closure there is a parallel connection of *R* and *L*:

$$i_L(t) + v(t)g = 0 \quad Lg\frac{di_L}{dt} + i_L = 0$$

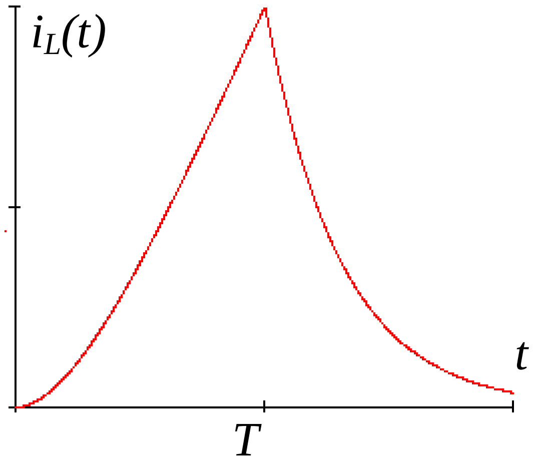
3) Characteristic equation is the same as previous.
4) As there is homogeneous equation, there is no particular solution at all.
5) Initial current $i_L(T+) = i_L(T-)$, thus integration constant is:

$$i_L(T) = i_L(T-) = B$$

6) Final formula is:

$$i_L(t) = i_L(T-)e^{-\frac{t-T}{Lg}}$$

Following figure shows the processes.



Solution representations

Mathematics gives us formal representation of the ODE total solution – particular solution of non-homogeneous ODE and general solution of the homogeneous ODE. Physicists and engineers prefer other representations of an ODE solutions.

First one is a sum of **<u>natural solution</u>** and **<u>forced solution</u>**. Natural solution corresponds to the initial conditions and may be considered as a process caused by *initial energy* stored in elements (inductances and capacitances) *without sources* of energy (independent e.f.m. and current sources). Forced solution corresponds to *sources*, but with *zero initial conditions*.

Our first example (charge of a capacitor) contains forced solution only. Second example (discharge of capacitor) contains a natural solution only. Note that natural solution may be considered as a part of the general solution, because both of them correspond to homogeneous

ODE. Forced solution always contains part of general solution and whole particular solution (in other words, particular solution is a part of forced solution).

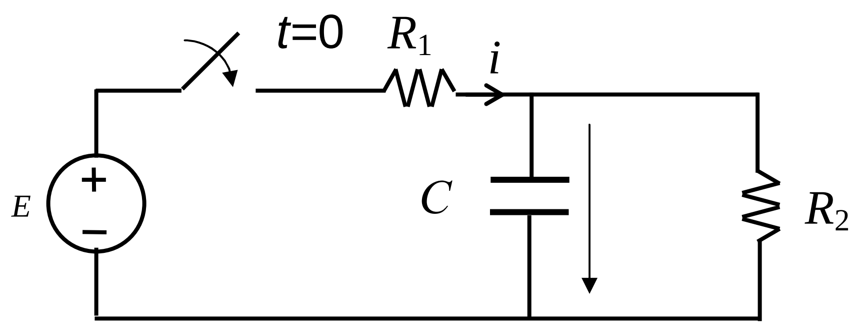
The second representation is the sum of the **steady-state solution** and **transient solution**. Steady-state solution is the same as particular solution of an ODE when there are DC, AC or periodical sources. In other words, steady-state solution may be obtained by means of *harmonic analysis*. Transient solution corresponds to a general solution of ODE. Our first example has both steady-state and transient parts (particular and general solution respectively); second example includes transient solution only and third example (with inductance) has transient solution only (ramp source isn't periodical, AC or DC source, so there is no steady state at all).

Solution without equation

Formal division of solution by types seems to be silly, but it has great positive effect. If there is steady-state in the circuit after the switcher closure, it is possible to obtain a solution of a problem without differential equation. There are two hints:

1) time constant of the *RC* loop is equal to $\tau = RC$, of *RL* loop – $\tau = L/R$. If there is more sophisticated circuit, it is necessary to set all independent sources to zero and to find equivalent resistance (input resistance) that is connected to *C* or *L*.
2) steady-state solution may be found by using DC analysis for DC sources, AC analysis for AC sources, harmonic analysis for other periodical sources and superposition theorem.

Let's consider an example:

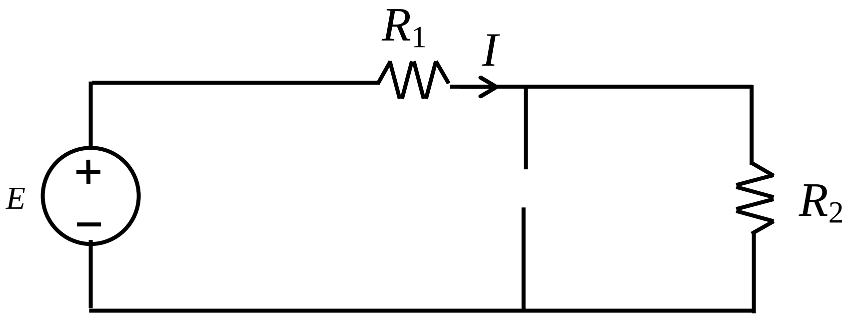


1) $v_C(0\text{-}) = 0$ as there no sources in the circuit before switcher closure;
2) equivalent resistance (or input resistance) is:

$$R = R_1 \parallel R_2 = \frac{R_1 R_2}{R_1 + R_2}$$

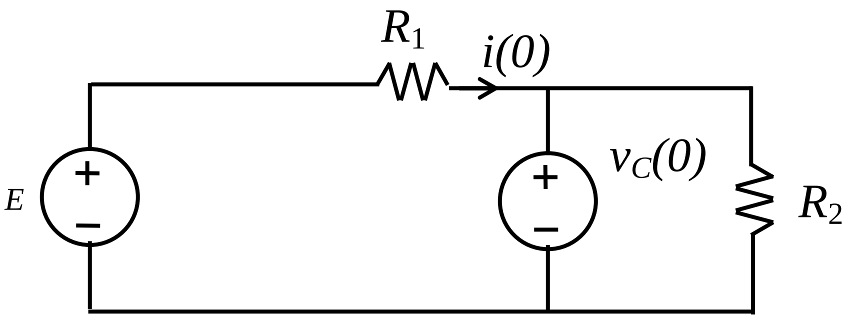
$$\tau = RC = C\frac{R_1 R_2}{R_1 + R_2}$$

3) steady-state (DC) solution (note that capacitance is substituted by the open circuit in steady-state):



$$I = \frac{E}{R_1 + R_2}$$

4) initial condition may be found in the scheme for the moment $t = 0+$:



$$i(0) = \frac{E - v_C(0)}{R_1} = \frac{E - 0}{R_1}$$

And, finally, integrating constant is:

$$i(0) = I + A$$

$$A = \frac{E}{R_1} - \frac{E}{R_1 + R_2}$$

Thus, result is:

$$i(t) = I + Ae^{-\frac{t}{\tau}} = \frac{E}{R_1 + R_2} + \frac{ER_2}{R_1(R_1 + R_2)} e^{-\frac{t(R_1 + R_2)}{CR_1R_2}}$$

Conclusion

Dynamics of the electrical circuit is described as a linear ordinary differential equation with constant coefficients. ODE formulation is the most difficult part of the time-domain analysis. If there is a steady-state model applicable, one can obtain total solution without ODE formulation. Usually, it is necessary to use steady-state analysis twice (before and after switcher closure). Initial condition may be obtained by the DC analysis, compensation principle and commutation laws. Root of the characteristic equation may be found through time constant of elementary circuit – it is just necessary to find input resistance.

Lecture 14.

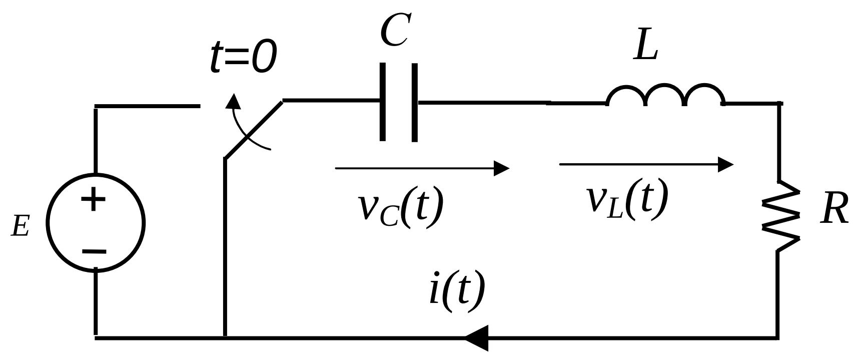
Transient analysis. Second order ODE. Types of natural solutions.

Introduction

This lecture continues previous topic – time domain analysis. We've studied techniques of the first order linear ordinary differential equation (ODE) with constant coefficients solution. Note that we could obtain all expressions in analytic form – as function of time with certain coefficients that have been connected with resistances and capacitance (or inductance). It will be shown that second order ODE requires *numerical evaluation* of the certain coefficients and common analytic formulas cannot be used.

Second order ODE

Let's consider process caused by the e.m.f. source attached to RLC loop (there is zero current in the inductance and zero voltage on capacitance *before* switcher closure):



KVL leads to the following expression:

$$-E+v_C+v_L+iR=0$$

Substituting voltage on inductance by the first derivative of the current, one obtains:

$$v_L=L\frac{di}{dt}$$

$$-E+v_C+L\frac{di}{dt}+iR=0$$

Then again, substituting current by the first derivative of the voltage on the capacitance, one can find non-homogeneous second order ODE for the v_C:

$$i=C\frac{dv_C}{dt}$$

$$-E+v_C+L\frac{d}{dt}C\frac{dv_C}{dt}+C\frac{dv_C}{dt}R=0 \quad\rightarrow\quad v_C+CR\frac{dv_C}{dt}+LC\frac{d^2v_C}{dt^2}=E$$

Differentiate it once and use constitutive relation to obtain homogeneous ODE of the second order for the current *i*:

$$\frac{d}{dt}v_C + CR\frac{d}{dt}\frac{dv_C}{dt} + LC\frac{d}{dt}\frac{d^2 v_C}{dt^2} = \frac{d}{dt}E \quad \rightarrow \quad \frac{1}{C}i + CR\frac{1}{C}\frac{di}{dt} + LC\frac{1}{C}\frac{d^2 i}{dt^2} = 0 \quad \rightarrow$$

$$\rightarrow \quad i + CR\frac{di}{dt} + LC\frac{d^2 i}{dt^2} = 0$$

Differentiate it once more and use the constitutive relation to obtain homogeneous ODE for the voltage v_L:

$$\frac{d}{dt}i + CR\frac{d}{dt}\frac{di}{dt} + LC\frac{d}{dt}\frac{d^2 i}{dt^2} = 0 \quad \rightarrow \quad \frac{1}{L}v_L + CR\frac{1}{L}\frac{dv_L}{dt} + LC\frac{1}{L}\frac{d^2 v_L}{dt^2} = 0 \quad \rightarrow$$

$$\rightarrow \quad v_L + CR\frac{dv_L}{dt} + LC\frac{d^2 v_L}{dt^2} = 0$$

All three ODEs have same coefficients in left hand. It means that there is the same (the only!) characteristic equation for all three processes. Note that there are two homogeneous ODEs with the same coefficients for i and v_L. Solutions of the same ODEs will be different – because of different initial conditions.

Particular solution for the voltage v_C may be found by the substitution of the v_{CP} by the constant V (as right hand contains just a constant) or through the steady-state analysis. Anyway, $v_{CP}(t) = E$.

General solution

First of all, the characteristic equation is:

$$1 + \lambda CR + \lambda^2 LC = 0 \quad \rightarrow \quad \lambda^2 + \lambda\frac{R}{L} + \frac{1}{LC} = 0 \quad \rightarrow \quad \lambda^2 + \lambda\frac{\omega_0}{Q} + \omega_0^2 = 0$$

We've already seen these coefficients –

$\frac{1}{\sqrt{LC}}$ - is resonance angular frequency ω_0;

$\frac{1}{R}\sqrt{\frac{L}{C}}$ is quality factor Q (see the lecture 11).

Thus, characteristic equation may be rewritten in form:

$$\lambda^2 + \lambda\frac{\omega_0}{Q} + \omega_0^2 = 0$$

One can use commonly known technique of the square equation solution through discriminant evaluation:

$$D = \frac{\omega_0^2}{Q^2}4\omega_0^2 = \omega_0^2\left(\frac{1}{Q^2} - 1\right)$$

There are three possible cases:

$D > 0$ (when $Q < 0.5$) – there are two *different real* roots:

$$\lambda_1 = -\frac{\omega_0}{2Q} - \frac{\omega_0}{2}\sqrt{\left(\frac{1}{Q^2} - 1\right)}$$

$$\lambda_2 = -\frac{\omega_0}{2Q} + \frac{\omega_0}{2}\sqrt{\left(\frac{1}{Q^2} - 1\right)}$$

general solution is:

$$v_{CG} = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

This process is called **overdamped process**.

$D = 0$ (when $Q = 0.5$) – there are two *equal real* roots:

$$\lambda_1 = \lambda_2 = -\frac{\omega_0}{2Q}$$

general solution is:

$$v_{CG} = Ae^{\lambda_1 t} + Bte^{\lambda_1 t}$$

This process is called **critically damped process**.

$D < 0$ (when $Q > 0.5$) – there is a pair of complex-conjugated roots:

$$\lambda_1 = \lambda_2^* = -\frac{\omega_0}{2Q} + j\frac{\omega_0}{2}\sqrt{1 - \frac{1}{Q^2}} = \alpha + j\omega_N$$

general solution is:

$$v_{CG} = Ae^{-\alpha t}\sin(\omega_N t + \varphi)$$

This process is called **underdamped process**; ω_N is called **frequency of natural oscillations**; α is called **damping constant**.

Note that it is necessary to *evaluate* roots of the characteristic equation to determine general solution type!

Another important note is that each type of the general solution consists of two integration constants (*A* and *B* for overdamped and critically damped processes or *A* and φ for underdamped one). Of course, two integration constants correspond to the pair of initial conditions:

- first initial condition is an initial value of the our function (e.g. $v_C(0)$);
- second initial condition is an initial value of the *first derivative* of the our function (e.g. $v_C'(0)$).

Finding initial conditions

The *classic* way of the initial conditions finding is based on Kirchhoff’s laws and constitutive relations – as an ODE formulation.

$v_C(0) = 0$ and $i(0) = 0$ – according to commutation law;

$$\left.\frac{dv_C}{dt}\right|_0 = \frac{i(0)}{C} = 0$$

- according to constitutive relation;

and, finally, it is necessary to differentiate KVL expression to determine first derivative of the inductance voltage:

$$-E + v_C(0) + v_L(0) + i(0)R = 0 \quad \to \quad v_L(0) = E - i(0)R - v_C(0) = E$$

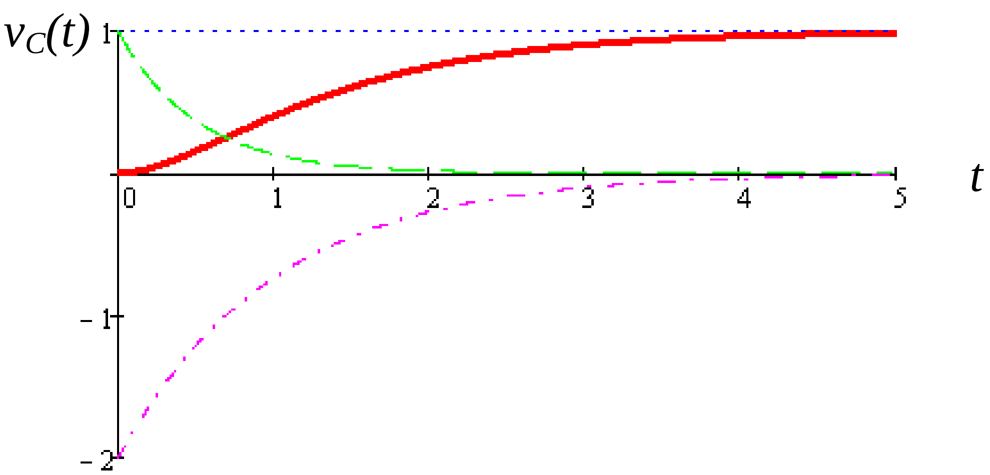
$$\left.\frac{di}{dt}\right|_0 = \frac{v_L(0)}{L} = \frac{E}{L}$$

$$-\frac{d}{dt}E + \frac{d}{dt}v_C + \frac{d}{dt}v_L + \frac{d}{dt}iR = 0 \quad \to \quad \left.\frac{dv_L}{dt}\right|_0 = \left.\frac{dv_C}{dt}\right|_0 - R\left.\frac{di}{dt}\right|_0 = -\frac{ER}{L}$$

Then, integration constants may be found. Following figures show processes of different type:

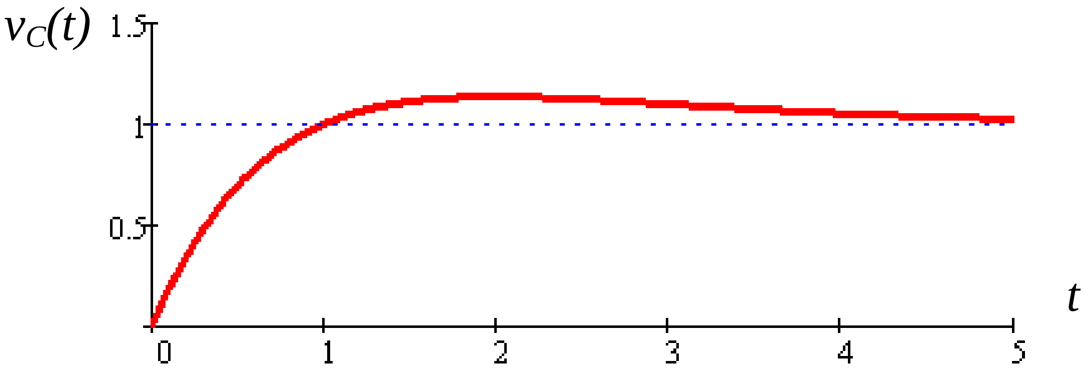
1) for overdamped process:

$$\begin{cases} v_C = E + Ae^{\lambda_1 t} + Be^{\lambda_2 t} \\ \dfrac{dv_C}{dt} = A\lambda_1 e^{\lambda_1 t} + B\lambda_2 e^{\lambda_2 t} \end{cases} \to \begin{cases} v_C(0) = E + A + B \\ \left.\dfrac{dv_C}{dt}\right|_0 = A\lambda_1 + B\lambda_2 \end{cases}$$



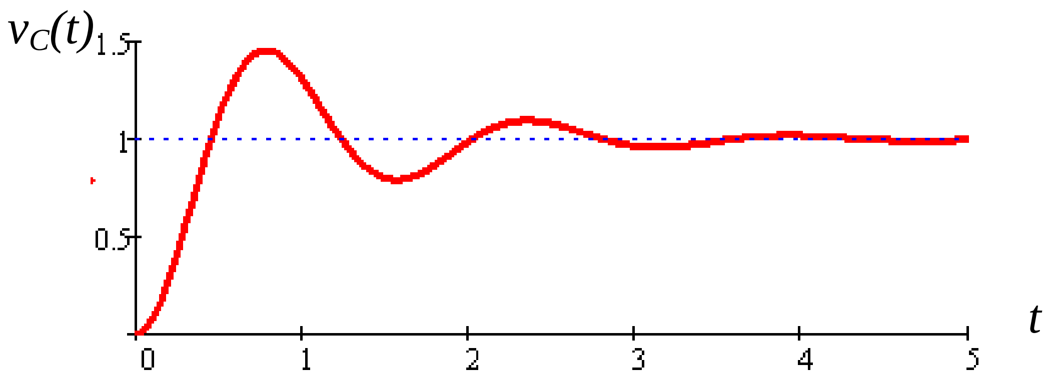
2) for critically damped process:

$$\begin{cases} v_C = E + Ae^{\lambda_1 t} + Bte^{\lambda_1 t} \\ \dfrac{dv_C}{dt} = A\lambda_1 e^{\lambda_1 t} + Bt\lambda_2 e^{\lambda_1 t} + Be^{\lambda_1 t} \end{cases} \to \begin{cases} v_C(0) = E + A \\ \left.\dfrac{dv_C}{dt}\right|_0 = A\lambda_1 + B \end{cases}$$

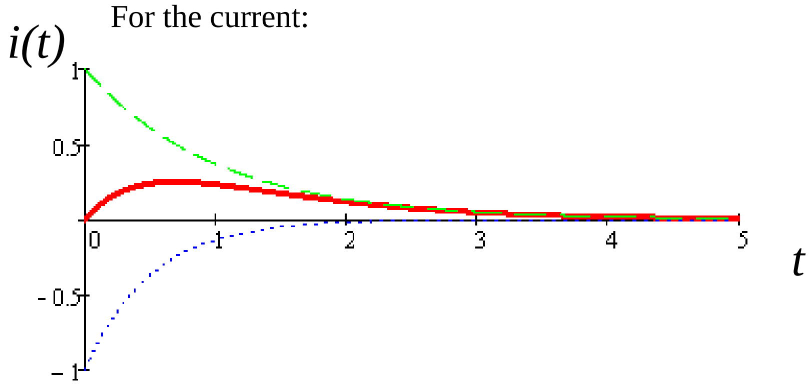


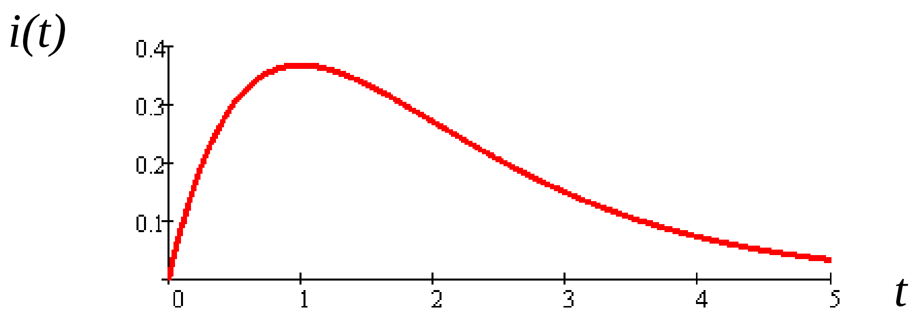
3) for underdamped process:

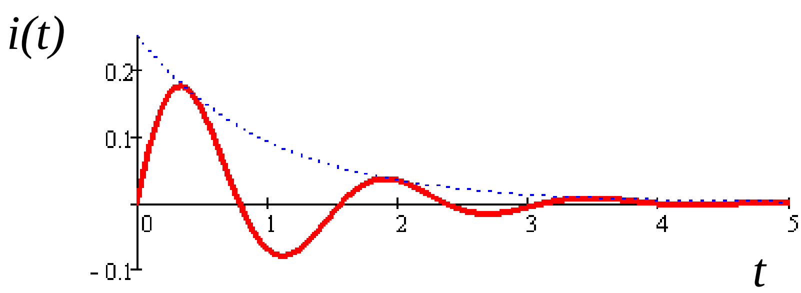
$$\begin{cases} v_C = E + Ae^{-\alpha t}\sin(\omega_N t + \varphi) \\ \dfrac{dv_C}{dt} = -\alpha Ae^{-\alpha t}\sin(\omega_N t + \varphi) + \omega_N Ae^{-\alpha t}\cos(\omega_N t + \varphi) \end{cases} \rightarrow \begin{cases} v_C(0) = E + A\sin(\varphi) \\ \left.\dfrac{dv_C}{dt}\right|_0 = -\alpha A\sin(\varphi) + \omega_N A\cos(\varphi) \end{cases}$$



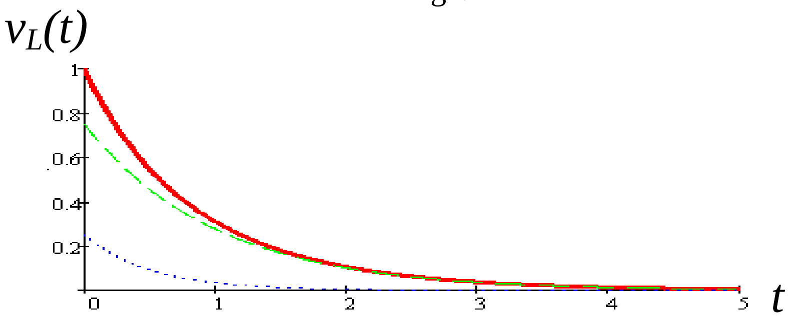
For the current:

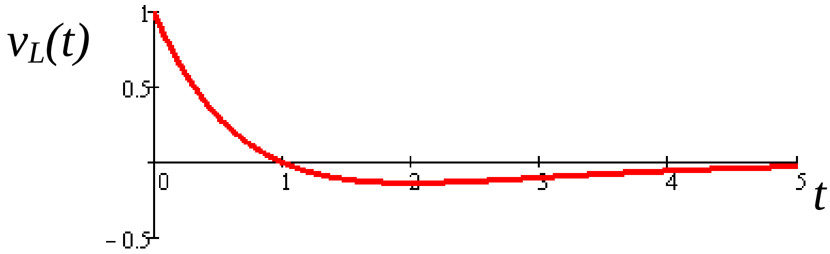


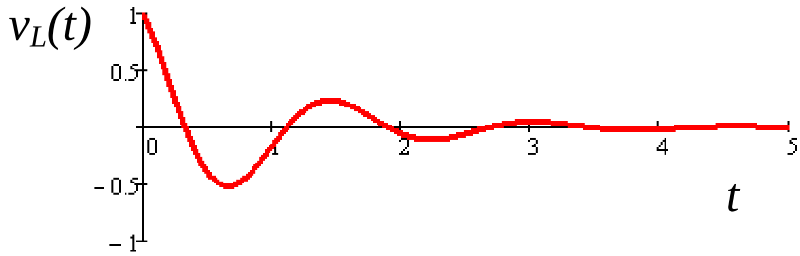




For the inductance voltage:







Common case – high order ODE

It might be misunderstood that each high order circuit had new, original form of the general (natural) solution. In fact, tree type of processes that are listed above cover all possible cases. As coefficients of a high-order ODE are formed by positive real constants (resistances, capacitances, inductances), the algebraic characteristic equation may have just three types of the roots – *different* or *equal real* valued and *pairs* of *complex-conjugated* roots.

For example, eighth order ODE

$$b_8\frac{d^8 i}{dt^8}+b_7\frac{d^7 i}{dt^7}+b_6\frac{d^6 i}{dt^6}+b_5\frac{d^5 i}{dt^5}+b_4\frac{d^4 i}{dt^4}+b_3\frac{d^3 i}{dt^3}+b_2\frac{d^2 i}{dt^2}+b_1\frac{di}{dt}+b_0 i = f(t)$$

has eight roots of characteristic equation:

$$b_8\lambda^8+b_7\lambda^7+b_6\lambda^6+b_5\lambda^5+b_4\lambda^4+b_3\lambda^3+b_2\lambda^2+b_1\lambda+b_0 i=0 \quad\rightarrow\quad \lambda_1,\lambda_2,\ldots,\lambda_8$$

General solution of this equation will be written as:

$$i_G(t)=A_8e^{\lambda_8 t}+A_7e^{\lambda_7 t}+\left(A_6e^{\lambda_6 t}+A_5+A_4t+A_3t^2\right)e^{\lambda_3 t}+A_2e^{-\alpha t}\sin\left(\omega_N t+\varphi\right)$$

To find all eight integration constants it is necessary to use eight initial conditions:

$$\left.\frac{d^7 i}{dt^7}\right|_0,\quad \left.\frac{d^6 i}{dt^6}\right|_0,\quad \left.\frac{d^5 i}{dt^5}\right|_0,\quad \left.\frac{d^4 i}{dt^4}\right|_0,\quad \left.\frac{d^3 i}{dt^3}\right|_0,\quad \left.\frac{d^2 i}{dt^2}\right|_0,\quad \left.\frac{di}{dt}\right|_0,\quad i(0)$$

- initial value, first, second and others, up to seventh derivatives at $t = 0$.

It seems to be difficult to find as large number of initial conditions. Moreover, there are no effective methods of the high order algebraic equation roots extraction. One can use numerical methods, but the most efficient methods are based on matrix *eigenvalue* problem solution.

State-space model

Linear circuits and corresponded linear ODE of the high order allow another representation – *system of the first order differential equations* instead of one high-order ODE. It is complicated problem in a common case, but in the electrical engineering there is an easy way to form this representation.

State variables of the circuit (or any dynamic system) are the variables that allow to compute *any response* (or output signal) of the circuit in interval $t \geq t_0$ if

- all input signals (sources) are given for $t \geq t_0$
- initial values of the state variables are known at $t = t_0$

for any starting time t_0.

Voltages on the capacitances and currents in inductances form a set of the state variables in the electrical circuits.

Denote state variables as a vector (column) of the functions:

$$\bar{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

input signals as a vector of the functions:

$$\bar{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \cdots \\ v_m(t) \end{bmatrix}$$

vector of output signals:

$$\bar{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdots \\ y_k(t) \end{bmatrix}$$

State-space is a total set of all state variables of the circuit. **State of the circuit** is the certain value of the state variables vector. **State-space model** of the circuit is the pair of the matrix first order ODEs:

$$\begin{cases} \dfrac{d\,\bar{x}(t)}{dt} = \mathbf{A}_1\bar{x}(t) + \mathbf{B}_1\bar{v}(t) \\ \bar{y}(t) = \mathbf{A}_2\bar{x}(t) + \mathbf{B}_2\bar{v}(t) \end{cases}$$

where

$\boldsymbol{A}_1$ – square matrix $[n \times n]$ of the coefficients;

$\boldsymbol{B}_1$ – matrix $[n \times m]$ of the coefficients;

$\boldsymbol{A}_2$ –matrix $[l \times n]$ of the coefficients;

$\boldsymbol{B}_2$ –matrix $[l \times m]$ of the coefficients.

Solution of the state-space model may be obtained in general form through the matrix exponent:

$$\bar{x}(t) = e^{\mathbf{A}_1 t}\bar{x}(0) + e^{\mathbf{A}_1 t}\int_0^t e^{-\mathbf{A}_1 t}\mathbf{B}_1\bar{v}(t)dt$$

Matrix exponent may be evaluated with Backer’s formula:

$$e^{\mathbf{A}_1 t} = \frac{D_{n-1}}{D}\mathbf{A}_1^{\,n-1} + ... + \frac{D_2}{D}\mathbf{A}_1^{\,2} + \frac{D_1}{D}\mathbf{A}_1 + \frac{D_0}{D}\mathbf{1}$$

where D_i – Van der Mond’s determinants:

$$D = \begin{vmatrix} 1 & 1 & ... & 1 \\ \lambda_1 & \lambda_2 & ... & \lambda_n \\ \lambda_1^{\,2} & \lambda_2^{\,2} & ... & \lambda_n^{\,2} \\ ... & ... & ... & ... \\ \lambda_1^{\,n-1} & \lambda_2^{\,n-1} & ... & \lambda_n^{\,n-1} \end{vmatrix}, \; D_i = \begin{vmatrix} 1 & 1 & ... & 1 \\ \lambda_1 & \lambda_2 & ... & \lambda_n \\ ... & ... & ... & ... \\ \exp(\lambda_1 t) & \exp(\lambda_2 t) & ... & \exp(\lambda_n t) \\ \lambda_1^{\,i+1} & \lambda_2^{\,i+1} & ... & \lambda_n^{\,i+1} \\ ... & ... & ... & ... \end{vmatrix}$$

and λ_i – i-th eigenvalue of the matrix $\boldsymbol{A}_1$.

$$\det(\mathbf{A}_1 - \mathbf{1}\lambda) = 0 \quad \to \quad \lambda_j$$

These eigenvalues are equal to the roots of the characteristic equation of this circuit.

Matrices of coefficients may be obtained by simple DC analysis and superposition theorem. State-space models are used for analysis and synthesis of the large-scale systems (of *very* high order – 100 and up to 100000!). State-space analysis stays powerful tool for theoretical researches in the fields of electrical engineering, control systems design, cybernetics, system analysis, etc.

Conclusion

It is necessary to focus our attention on the fact that it is possible to discriminate type of the roots of the characteristic equation by the plot of the process – oscillations mark complex-conjugated roots. State-space model is also very useful formulation for the numerical analysis of the system dynamic. Moreover, similar approach is widely used in non-linear circuits as a base of phase plane representation in non-linear dynamics.

Lecture 15.

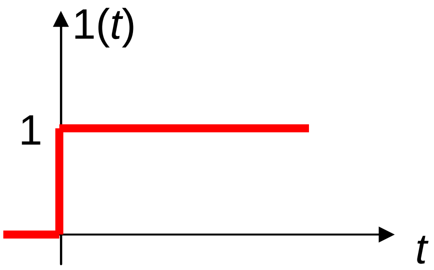
Advanced methods of transient analysis. s-domain analysis. Convolution.

Introduction

There are several special techniques are developed for ODE integration and adopted to the circuit analysis. These methods supplement state-space model and classical time domain analysis. Important advantage of such techniques is a kind of connection between time-domain dynamics and, for example, AC analysis, transfer functions and etc. Moreover, these techniques are, in fact, part of the modern engineer (researcher) language and it is necessary to their study properties to understand articles and books on different parts of engineering.

Special functions

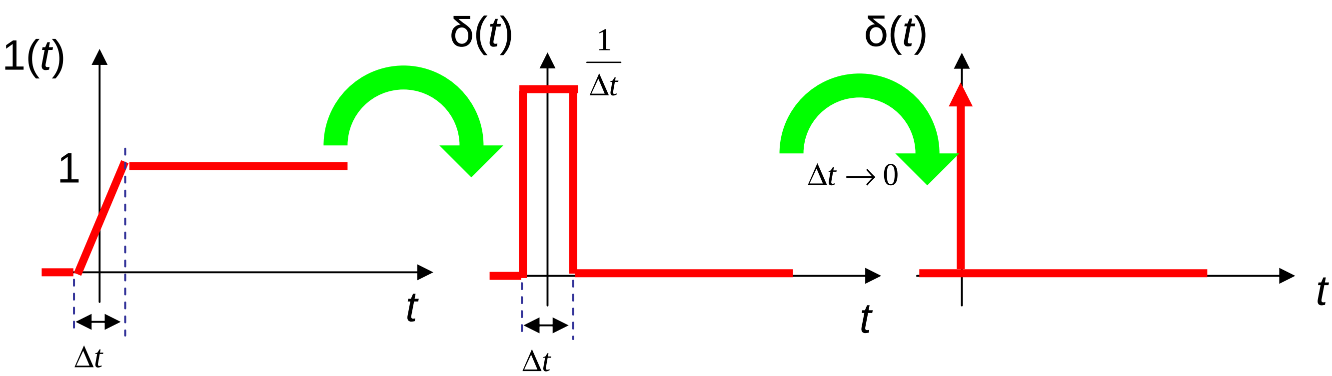
First of all, let's introduce pair of *special* functions: **unit step function** $1(t)$ (or **Heaviside function**, or just a **step function**) and **delta function** $\delta(t)$ (or **Dirac function**, or **delta impulse**). Step function is equal to zero when $t < 0$ and it is equal to 1 when $t \geq 0$:



$$1(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Delta function is introduced as first derivative of the unit step function with respect to time:

$$\delta(t) = \frac{d1(t)}{dt} = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$



So, it is equal to infinite at $t = 0$ and it is zero elsewhere! Note that square under this function is equal to 1:

$$S_\delta = \int_{-\infty}^{+\infty} \delta(t)dt = 1(t)\Big|_{-\infty}^{+\infty} = 1 - 0 = 1$$

These functions are widely used in the dynamic analysis in electrical engineering, digital data processing, control theory and etc. to model impulse sources uniformly.

Laplace transform

One of the most useful techniques is based on **Laplace transform**. Laplace transform is a formal representation of the time-domain function (or, in other words, of the function of time) in complex *s*-domain functional space:

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

$$F(s) = L(f(t))$$

s – complex variable; *f*(*t*) – original function of time; *F*(*s*) – Laplace representation of *f*(*t*).

Laplace transform is a *linear* transform:

$$F(s) = L(f(t)) \quad G(s) = L(g(t))$$

$$L(a \cdot f(t) + b \cdot g(t)) = aL(f(t)) + bL(g(t)) = aF(s) + bG(s)$$

Time-domain *derivative* is represented as an *algebraic* expression:

$$F(p) = L(f(t))$$

$$L\left(\frac{df(t)}{dt}\right) = L(f'(t)) = pF(p) - f(0)$$

Time-domain *integration* is also represented as an *algebraic* operation:

$$F(s) = L(f(t))$$

$$L\left(\int_0^t f(t) dt\right) = \frac{F(s)}{s}$$

If there is a *delayed* function, Laplace transform represent it as complex *exponent*:

$$F(s) = L(f(t))$$

$$L(f(t-\tau)) = F(s) e^{-s\tau}$$

Otherwise, *shift* of the s-domain representation corresponds to time-domain *exponential* function of the real argument:

$$F(s) = L(f(t))$$

$$L\left(f(t) e^{\alpha t}\right) = F(s-\alpha)$$

Time-domain function may be *scaled*:

$$F(s) = L(f(t))$$

$$L\left(\frac{1}{k} f\left(\frac{t}{k}\right)\right) = F(ks)$$

Convolution of two functions is an integral:

$$\int_0^t f(\tau)g(t-\tau)d\tau$$

Convolution has a Laplace representation as:

$$F(s)=L(f(t)) \quad G(s)=L(g(t))$$
$$L\left(\int_0^t f(\tau)g(t-\tau)d\tau\right)=F(s)G(s)$$

And there are *limit* properties:

$$F(s)=L(f(t))$$
$$f(0)=\lim_{s\to\infty}(sF(s))$$
$$f(\infty)=\lim_{s\to 0}(sF(s))$$

Laplace transform represent ODE as an algebraic equation in s-domain. So, it is introduced to help solving ODEs. For example, 2nd order ODE may be represented as:

$$i+CR\frac{di}{dt}+LC\frac{d^2 i}{dt^2}=0 \quad \to \quad I(s)+CR(sI(s)-i(0))+LC\left(s^2 I(s)-si(0)-\left.\frac{di}{dt}\right|_0\right)=0$$

And the Laplace representation of the *unknown* function is:

$$I(s)=\frac{CRi(0)+LC\left(si(0)+\left.\frac{di}{dt}\right|_0\right)}{1+sCR+s^2 LC}$$

Reverse Laplace transform

Reverse Laplace transform is denoted as:

$$f(t)=\frac{1}{2\pi j}\int_{-\sigma-j\infty}^{-\sigma+j\infty}F(s)e^{st}ds\,,\quad f(t)=L^{-1}(F(s))$$

It is a contour integral on the complex plane. Contour should be selected to be by the *left side* of the $F(s)$ **poles**. Pole of the complex function is a point at which this function approaches infinity (denominator of the $F(s)$ is equal to zero).

Let's considered $F(s)$ as a ratio of two polynomial functions $N(s)$ in the numerator, $D(s)$ in the denominator:

$$F(s)=\frac{N(s)}{D(s)}$$

Residue of the $F(s)$ at *pole* s_i is a ratio of $N(s)$ to $D(s)$, but multiplied by (s- s_i):

$$F(s)=\frac{\mathrm{res}(s_i)}{(s-s_i)}=\frac{N(s)}{D_i(s)(s-s_i)} \quad \to \quad \mathrm{res}(s_i)=\frac{N(s_i)}{D_i(s_i)}$$

Another way of the residue evaluation is obtained by Lopital's rule:

$$\mathrm{res}(s_i) = \lim_{s \to s_i}\left(\frac{N(s)}{D(s)}(s - s_i)\right) = \lim_{s \to s_i}\left(\frac{N'(s)(s - s_i) + N(s)}{D'(s)}\right) = \frac{N(s_i)}{D'(s_i)}$$

where *D'*(*s*) is a first derivative of the *D*(*s*) with respect to *s*.

Residue theorem leads us to the simple evaluation of the reverse Laplace transform:

$$\int_C F(s)ds = 2\pi j \ \mathrm{res}(s_i)$$

$$f(t) = \frac{1}{2\pi j}\int_{-\sigma - j\infty}^{-\sigma + j\infty} F(s)e^{st}ds = \frac{1}{2\pi j}\sum_{i=1}^{n} 2\pi j \, \mathrm{res}(s_i)e^{s_i t} = \sum_{i=1}^{n}\frac{N(s_i)}{D'(s_i)}e^{s_i t}$$

If there is a pair of the conjugated poles, residue theorem may be rewritten as:

$$f(t) = 2\,\mathrm{Re}\left[\frac{N(s_1)}{D'(s_1)}e^{s_1 t}\right]$$

If there are *n*-th order poles (there are *n* poles at the same point), residue theorem should be improved by the Lopital's rule.

One can use tables of the Laplace transform, for example, this one:

time-domain function	s-domain representation
1(t)	$\frac{1}{s}$
δ(t)	1
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$1 - e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$
$\frac{1}{\alpha - \beta}\left(e^{-\beta t} - e^{-\alpha t}\right)$	$\frac{1}{(s+\alpha)(s+\beta)}$
$\frac{1}{\alpha - \beta}\left(\beta e^{-\beta t} - \alpha e^{-\alpha t}\right)$	$\frac{s}{(s+\alpha)(s+\beta)}$
$\frac{1}{\alpha\beta} + \frac{1}{\alpha - \beta}\left(\frac{1}{\beta}e^{-\beta t} - \frac{1}{\alpha}e^{-\alpha t}\right)$	$\frac{1}{s(s+\alpha)(s+\beta)}$
$\cos\omega t$	$\frac{p}{p^2 + \omega^2}$
$\sin\omega t$	$\frac{\omega}{p^2 + \omega^2}$

Laplace transform and the state-space model

One can apply Laplace transform to a state-space model of the circuit:

$$\begin{cases} \dfrac{d\,\bar{x}(t)}{dt} = \mathbf{A}_1 \bar{x}(t) + \mathbf{B}_1 \bar{v}(t) \\ \bar{y}(t) = \mathbf{A}_2 \bar{x}(t) + \mathbf{B}_2 \bar{v}(t) \end{cases} \rightarrow \begin{cases} s\bar{X}(s) - \left.\dfrac{d\,\bar{x}(t)}{dt}\right|_0 = \mathbf{A}_1 \bar{X}(s) + \mathbf{B}_1 \bar{V}(s) \\ \bar{Y}(s) = \mathbf{A}_2 \bar{X}(s) + \mathbf{B}_2 \bar{V}(s) \end{cases}$$

It is just necessary to collect Laplace representation of the state variables vector on the left hand and divide right hand expression by the matrix:

$$\bar{X}(s) = (s - \mathbf{A}_1)^{-1} \left(\mathbf{B}_1 \bar{V}(s) + \bar{x}(0) \right)$$

It is common solution of the state-space equations. As a matrix exponent, this solution has relatively simple *form*, but it isn’t easy to evaluate this expression! In fact, this makes this model very attractive for theoretical issues – there is very short notation that conceals all the difficulties of certain circuit analysis.

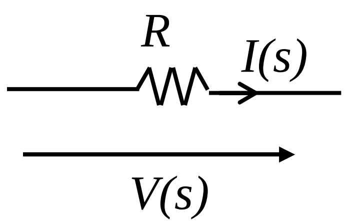
Operational method

Laplace transform allows to solve simple algebraic equation instead of ODE. The only disadvantage of this technique is that it is necessary to obtain differential equation – and it isn’t an easy to do. The **operational method** of the electrical circuit analysis was introduced by Oliver Heaviside. Advantage of this method is that there is no need to formulate differential equation – Laplace transform is applied to *elements* of the electrical circuit.

So that, each element has an equivalent operational scheme:

1) Resistance

Ohm’s law is a linear algebraic expression, so that it is valid for Laplace representations of the voltage and of the current:



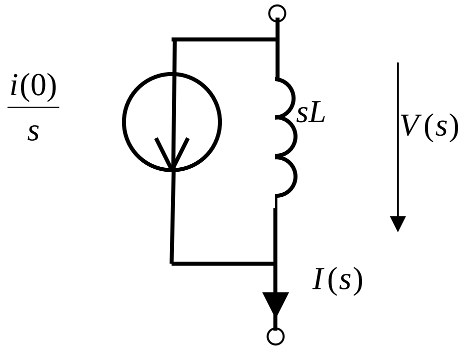
$$V(s) = R \cdot I(s),$$
$$I(s) = gV(s)$$

2) Inductance

Laplace transform is applied to a constitutive relation:

$$V(s) = sL \cdot I(s) - Li(0)$$
$$I(s) = \frac{V(s) + Li(0)}{sL} = \frac{V(s)}{sL} + \frac{i(0)}{s}$$

Equivalent operational scheme is shown on the following figure:



The sL expression is called **<u>operational impedance</u>** of the inductance.

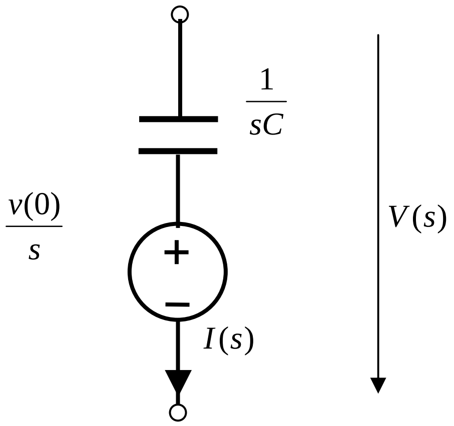
3) Capacitance

Laplace transform is applied to a constitutive relation:

$$I(s) = sC \cdot V(s) - Cv(0)$$

$$V(s) = \frac{I(s) + Cv(0)}{sC} = \frac{I(s)}{sC} + \frac{v(0)}{s}$$

Equivalent operational scheme is shown on the following figure:



The sC expression is called **<u>operational admittance</u>** of the capacitance.

Then, one can formulate the positions of the operational method:

1) Form an operational circuit – replace all elements by the equivalent schemes and use Laplace representations of the sources.
2) Find Laplace transform of the unknown functions – ordinary methods of DC analysis may be used.
3) Compute the original of the unknown function through reverse Laplace transform.

Step response and impulse response

There are two time-domain circuit characteristics – step response and impulse response. Their names determine their definitions:

<u>Step response</u> $h(t)$ is a reaction of the circuit caused by unit step (Heaviside) function;

<u>Impulse response</u> $w(t)$ is a reaction of the circuit caused by delta impulse (Dirac) function.

Each response is caused by the *one* specific function – so it is time-domain representation of a SISO system. There are *zero initial conditions* (or zero initial state) assumed for the response computation (or measurement).

If the input signal is associated with the voltage and output signal is associated with the voltage too, step response and impulse response are dimensionless functions. If the input signal is associated with the voltage and output signal is associated with the current, step response and impulse response are measured in Siemens and they are called **step transconductance** and ***impulse transconductance*** respectively. If the input signal is associated with the current and output signal is associated with the voltage, step response and impulse response are measured in Ohms and they are called **step transresistance** and **impulse transresistance** respectively. And, once again, if both input and output signals are associated with the currents, responses are dimensionless functions. But it is necessary to mention that input and output signals are usually associated with the voltages.

As the Laplace transform of the delta impulse is equal to 1, it is possible to show that Laplace transform of the impulse response is equal to transfer function of the circuit with the substitution $s = j\omega$:

$$W(s) = L\big(w(t)\big)$$

It is fundamental principle of the time-domain and frequency-domain representations equality. Linear SISO system may be perfectly represented by the transfer function or by the impulse response – each representation gives full information about system behavior.

Unit step function has the Laplace representation as s^{-1}. Thus step response may be computed as reverse Laplace transform of $s^{-1}W(s)$:

$$h(t) = L^{-1}\left(\frac{W(s)}{s}\right)$$

One can apply limit properties of the Laplace transform to obtain:

$$h(0) = \lim_{s\to\infty}\big(W(s)\big)$$

$$h(\infty) = \lim_{s\to 0}\big(W(s)\big)$$

Convolution, Duhamel's integral

One can apply convolution property of the Laplace transform to the transfer function definition:

$$W(s) = \frac{V_{out}(s)}{V_{in}(s)} \quad \rightarrow \quad V_{out}(s) = V_{in}(s)W(s) \quad \rightarrow \quad v_{out}(t) = \int_0^t v_{in}(x)w(x-t)dx$$

Thus, *s*-domain representation leads to the integral expression in time-domain. It may be shown that step response may be used in a similar integral expression – **Duhamel's integral**:

$$v_{out}(t) = v_{in}(0)h(t) + \int_{0+}^{t} v'_{in}(x)h(x-t)dx$$

In fact, it is convolution of the step response and the first derivative of the input voltage. These integral expressions are very useful for the data processing (digital and analog).

Conclusion

There are many special techniques of the system dynamic analysis (time-domain, *s*-domain, frequency-domain, functional-domain and etc.). One can mention diacoptic technique, wavelet transform, Gilbert transform. But most of all published papers, articles, textbook on electrical engineering, system analysis, control theory, computer sciences use s-domain analysis, state-space model and convolution technique.

Lecture 16.

Introduction in non-linear circuits

Introduction

Previous lectures described different methods of the *linear* circuit analysis. This lecture introduces several techniques of the *non-linear* analysis. There are many differences but the main is that there are no common, uniform, reliable methods of the non-linear problems solution. Each non-linear problem (dynamic or even static) requires unique, special technique. For example, we studied several methods of the high-order linear ODE integration (one can use classic approach, state-space method or Laplace transform) – they all are valid for every linear ODE, for every linear circuit; non-linear circuit of one type requires solution of Bernoulli equation, another circuit leads to Abel equation – there are completely different solution techniques. Moreover, there are no explicit solutions available for the most of non-linear circuits; it is necessary to use numerical methods (e.g., Newton-Raphson method).

Non-linear elements and their characteristics

Non-linear element is the electrical element that is described by the non-linear (algebraic or differential) equation.

Non-linear *algebraic* equation corresponds to *static* characteristic – non-linear resistance, non-linear controlled e.m.f. or current source. Static characteristic of the non-linear resistance is a well-known volt-amps diagram.

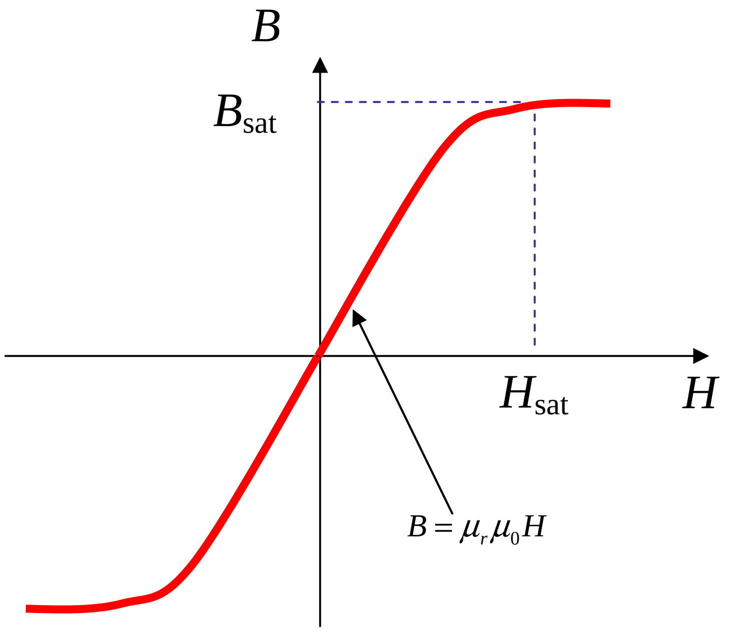
There are three ways of the non-linear volt-amps characteristic description:

1) table of pairs *V* and *I* – suitable for numerical analysis;
2) volt-amps diagram plot – suitable for graphical methods;
3) functional approximation of the *V*(*I*) or *I*(*V*) dependence – useful for analytic methods.

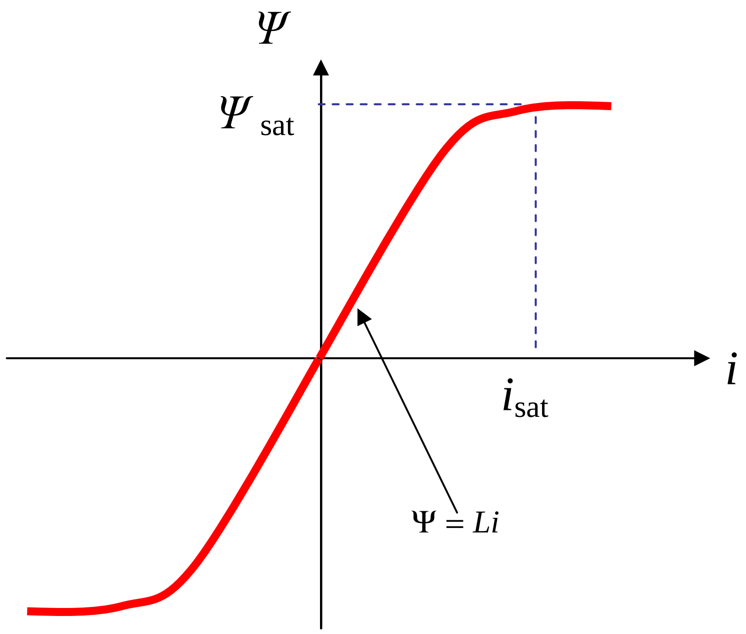
Non-linear resistance may be considered as a model of the energy absorption process that depends of the different physical parameters – ordinary electric bulb has non-linear characteristic (as conductivity of the wolfram glower depends on the temperature and temperature is determined by the passing current); semiconductor diode has non-linear volt-amps diagram (as an ability to conduct current is proportional to the exponential function of the voltage).

Real operational amplifier may be described as an example of non-linear controlled voltage source (as output voltage is limited by the power supply source).

If there is a non-linear differential expression – there is a non-linear dynamic characteristic. In fact, many of useful dynamic characteristics are given through static dependencies – non-linear inductance may be determined by the magnetization curve of the material of the core:



This curve determines non-linear dependence of the flux linkage on the current:



Thus, there will be non-linear dynamic equation for the voltage:

$$v(t,i) = \frac{d\Psi(t,i)}{dt}$$

Same thing may be said about non-linear capacitance – if there is non-linear volt-coulomb characteristic, there will be non-linear dynamic equation:

$$i(t,v) = \frac{dQ(t,v)}{dt}$$

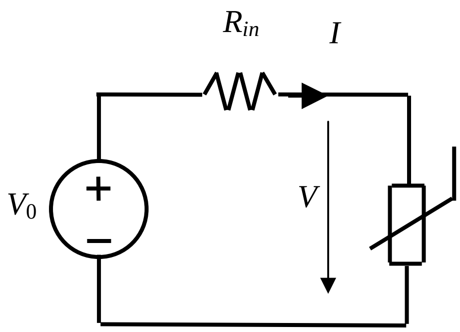
Important notes

Superposition principle is *unavailable* in non-linear circuits. Response of the non-linear circuit caused by several simultaneous stimuli *is not* equal to sum of the responses caused by each stimulus separately. Therefore, all techniques that are based on linear transform (e.g. Laplace transform) or summation (e.g. Fourier series) are, generally speaking, unavailable.

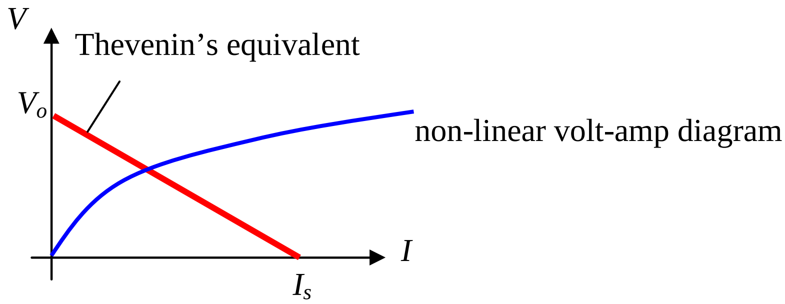
DC analysis of non-linear circuit – graphic approach

DC analysis assumes that all current and voltages are constants. In the non-linear circuits it is not enough to specify constant sources as a condition of the steady state! It may be necessary (and it may difficult!) to *prove* that there is steady state in the non-linear circuit.

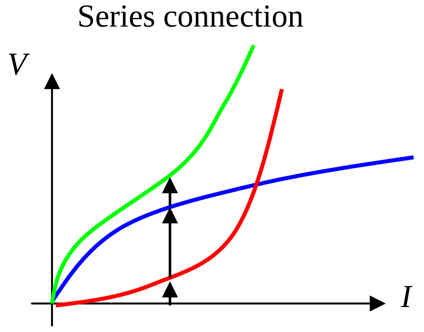
If there is the only non-linear element in the circuit, one can use Tevenin's or Norton's equivalent scheme for the rest (and linear) part of the circuit:

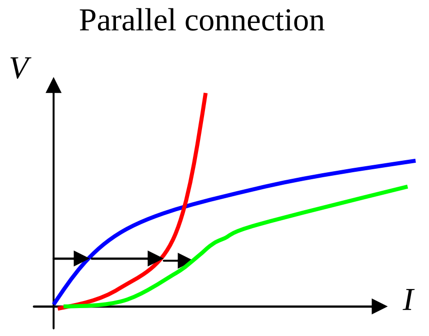


Solution may be obtained as an intersection of non-linear volt-amps diagram and linear characteristic of the equivalent scheme:



If there are several (more than one) non-linear elements, it may be helpful to combine volt-amps diagrams of the parallel or series connection of elements:





Parallel connection requires to sum currents of elements at the same voltage. Series connection requires to sum voltages at same current. It is not guaranteed that it is possible to combine all elements – at almost all circuits contain non-series and non-parallel connections.

DC analysis of non-linear circuit – analytic method

First step of the analytic method is an *approximation* of non-linear volt-amps diagrams by the suitable functions. The best choice is a polynomial function as it leads to non-linear polynomial equations. Other functions (e.g. exponential or logarithmic) lead to the *transcendental* equations that are much more difficult to solve.

Second step is the system of equations formulation based on KVL, KCL, mesh analysis or nodal analysis. If there is the only non-linear element in the circuit, one can use Tevenin's or Norton's equivalent scheme for the rest (and linear) part of the circuit. Then the only non-linear equation will be formed:

$$V(I) - V_0 + IR_{in} = 0 \quad \leftrightarrow \quad I(V) - I_S + Vg_{in} = 0$$

Note that polynomial equation has the as many roots as an *order* of the equation is (the order of equation is determined by the order of polynomial approximation). Transcendental equation may have *infinite* number of roots. Each root of the non-linear algebraic equation corresponds to *possible* steady state of the circuit! So it is necessary:

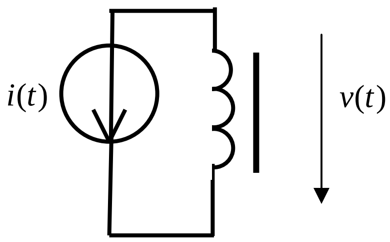
1) check approximation – function should *fit* volt-amps diagram at roots of the equation as good as it possible, otherwise one can reject certain root as non-physical;
2) if there are several possible solutions, the only way to specify one of them as *real* steady state is the dynamic analysis. One can find that certain initial state leads to certain steady state.

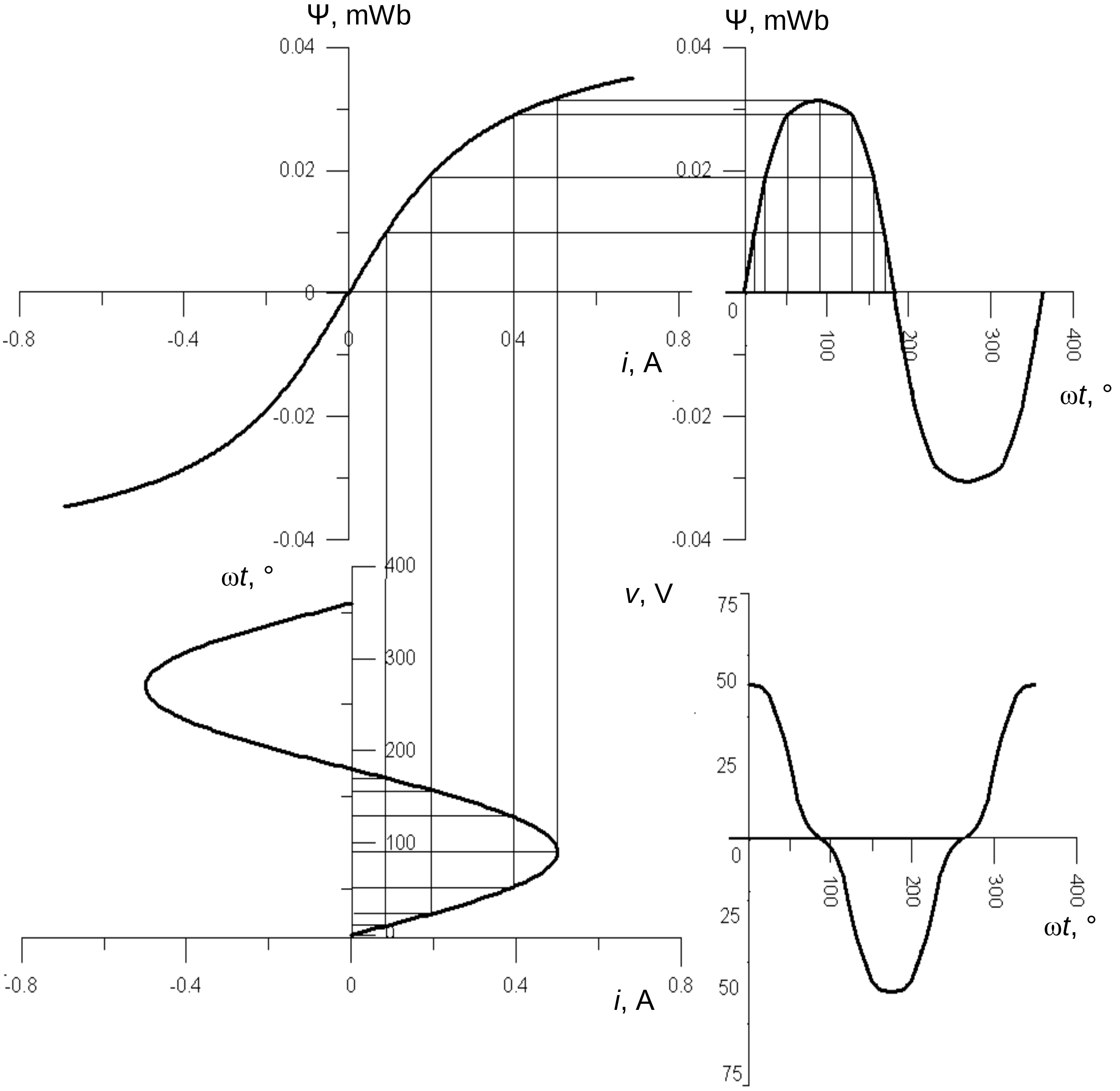
AC analysis of non-linear circuit – graphical approach

AC analysis of non-linear circuit deals with sinusoidal sources. It must be underlined that all other currents and voltages in the non-linear circuit *are not sinusoidal*.

Therefore, *complex* approach is *unavailable*, as well as harmonic analysis.

Graphical approach may be used for very simple circuits only. For example, it is possible to compute voltage on non-linear inductance, when sinusoidal current is given:



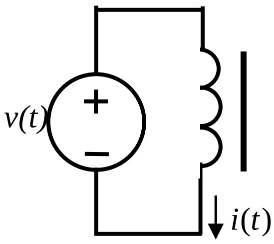


Note that there are pulses of the voltage, so one can conclude that it is possible to use non-linear inductor as sinusoidal input to pulse output adapter.

AC analysis of non-linear circuit – analytic method

Of course, analytic method is available for relatively simple circuits. It deals with non-linear approximation and refers to non-linear dynamics in most of all practical cases.

Let's consider non-linear inductance attached to sinusoidal e.m.f. source:



$B(H)$ dependence (magnetization curve) is given as 3rd order (cubic) polynomial function:

$H(B) = aB^3 + bB$

Magnetic flux may be obtained as:

$$\Psi(t) = \int_0^t v(t)dt = \int_0^t V_a \cos(\omega t)dt = \frac{V_a}{\omega}\sin(\omega t)$$

Then, magnetic flux density is (w - number of turns, S - square of the magnetic core):

$$B(t) = \frac{\Psi(t)}{Sw} = \frac{V_a}{Sw\omega}\sin(\omega t)$$

Polynomial approximation gives:

$$H(t) = a\left(\frac{V_a}{Sw\omega}\right)^3 \sin^3(\omega t) + b\frac{V_a}{Sw\omega}\sin(\omega t) = \left[b\frac{V_a}{Sw\omega} + \frac{3a}{4}\left(\frac{V_a}{Sw\omega}\right)^3\right]\sin(\omega t) - \frac{a}{4}\left(\frac{V_a}{Sw\omega}\right)^3 \sin(3\omega t)$$

Magnetic field H corresponds to a current (l - effective length of the core):

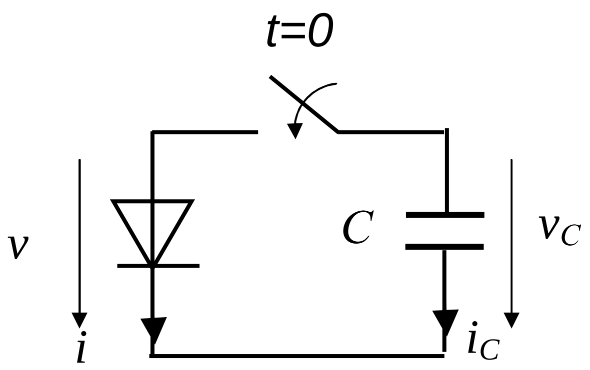
$$i(t) = \frac{H(t)}{w}l = \left[b\frac{V_a l}{Sw^2\omega} + \frac{3a}{4w}l\left(\frac{V_a}{Sw\omega}\right)^3\right]\sin(\omega t) - \frac{al}{4w}\left(\frac{V_a}{Sw\omega}\right)^3 \sin(3\omega t)$$

Note that current doesn't have sinusoidal waveform – it contains sinusoidal function of different frequencies – 1st and 3rd harmonics. First harmonic corresponds to a sinusoidal e.m.f. source. Third harmonic is derived from cubic approximation. It is evident that 5th order polynomial approximation leads to 3rd and 5th harmonics; 7th order approximation - to 3rd, 5th and 7th harmonics. Even order of polynomial approximation doesn't fit magnetization curve – it is odd function. But the polynomial approximation is just an our choice, the more precise approximation is obtained by hyperbolic tangent function. Of course, it is odd function and it leads to the *infinite* number of *odd* harmonics. This simple problem is, in fact, valuable – if circuit contains a non-linear inductance (really, any inductor with a core), the *odd* harmonics of the current (and voltage) are expected.

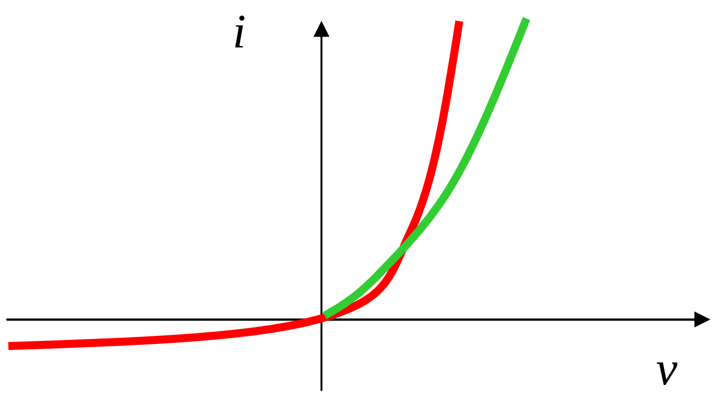
Non-linear dynamics

Non-linear dynamics is based on the non-linear ODEs of different type. There are no common techniques of the non-linear ODE integration, so in most cases numerical analysis is preferable. Analytic solution is available for very few simple problems. Powerful method of the non-linear dynamic analysis is based on the **piecewise linear approximation**.

Let's consider simple example – capacitance discharge through the semiconductor diode:



Analytic solution may be obtained for 2nd order polynomial approximation of the diode volt-amps diagram:



$i = av^2$

KVL gives non-linear differential equation:

$$i = -i_C = -C\frac{dv}{dt} \quad \rightarrow \quad av^2 = -C\frac{dv}{dt}$$

It is possible (in this particular case) to integrate left hand and right hand expressions separately:

$$-\frac{adt}{C} = \frac{dv}{v^2} \quad \rightarrow \quad -\int\frac{a}{C}dt = \int\frac{dv}{v^2}$$

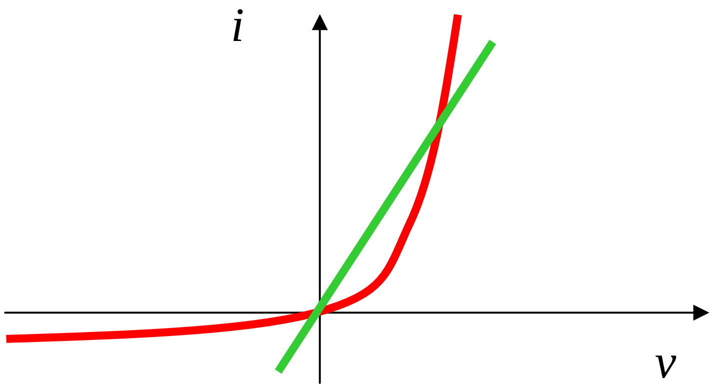
$$-\frac{at}{C} + k = -\frac{1}{v} \quad \rightarrow \quad v(t) = \frac{C}{a(t+k')}$$

Thus, applying initial voltage at $t = 0$ one can obtain:

$$v(0) = v_C(0) = \frac{C}{ak'} \quad \rightarrow \quad k' = \frac{C}{a v_C(0)}$$

$$v(t) = \frac{C}{a\left(t + \frac{C}{a v_C(0)}\right)} = \frac{v_C(0)}{\left(t\frac{a v_C(0)}{C} + 1\right)} = \frac{v_C(0)}{\left(t/\tau + 1\right)} \quad \tau = \frac{C}{a v_C(0)}$$

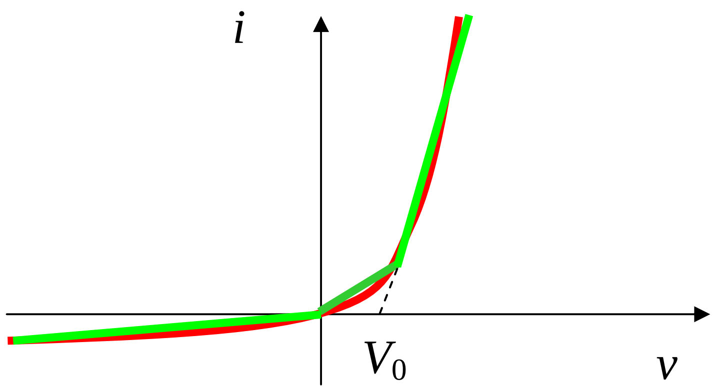
Linearization (diode volt-amps diagram is replaced by the linear function, so diode is replaced by the resistance *R*):



This situation is rather simple, voltage on the capacitance may be found as:

$$v(t) = v_C(0)e^{-t/\tau} \quad \tau = RC$$

Piecewise linear approximation (three linear sections are chosen):



First section corresponds to equation:

$$v = R_1 i$$

- it is same as Ohm's law, so diode may replaced by resistance.

Second section is also simple function:

$$v = R_2 i$$

$$v \le 0{,}7 \quad R_2 = 200$$

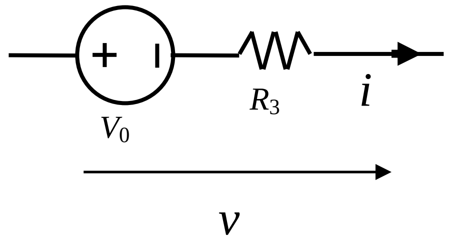
- diode may be replaced by *another* resistance.

Third section corresponds to the following equation:

$$v = V_0 + R_3 i$$

$$v > 0{,}7, \quad R_3 = 25, \quad V_0 = 0{,}6125$$

Equivalent scheme of the diode consists of resistance and e.m.f. source:



As initial voltage $v(0)$=1V is greater than upper bound of the 2^{nd} section, we start solution with the third section:

Steady state voltage in this scheme is equal to V_0.

Time constant is: R_3C

Voltage may be found as:

$$v(t) = V_0 + \left(v(0) - V_0\right)e^{-\frac{t}{R_3C}} = 0{,}6125 - 0{,}0875e^{-\frac{t}{R_3C}}$$

At the moment $t = t_0$, voltage drops below the upper bound of the second section:

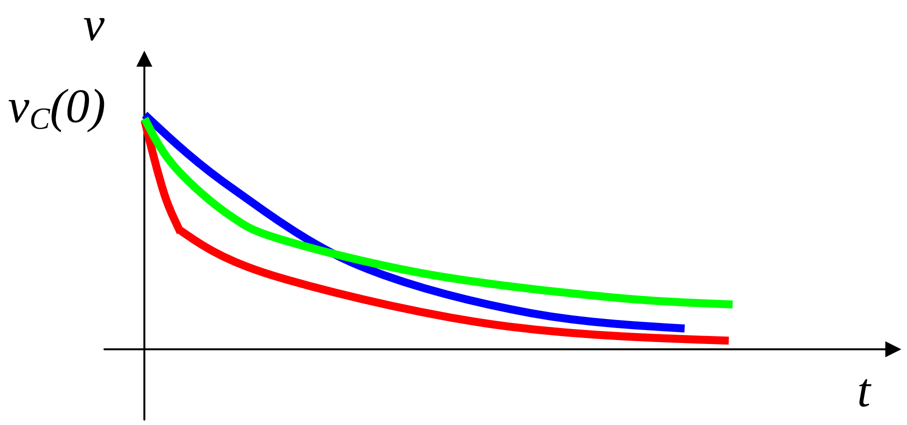
$$v(t_0) = 0{,}7 = 0{,}6125 - 0{,}0875e^{-\frac{t_0}{R_3C}}$$

Then, the second section becomes active:

$$v(t) = v(t_0)e^{-\frac{t_0}{R_2C}} = 0{,}7e^{-\frac{t_0}{R_2C}}$$

(with initial voltage $v(t_0)$)

Plots of processes computed by three different methods are show on the following figure:



quadratic approximation - green, linearization - blue, piece-wise approximation - red

Important note: there is possible situation when there is no steady state in the non-linear circuit with the DC sources (chaotic mode) or there is sinusoidal (triangular, rectangular or other periodical pulses) output signal caused by the DC input.

Conclusion

It is possible to say that non-linear analysis is too difficult for us and it is possible to exclude this lecture out of scope of our course, but there is an important reason to (at least) introduce basic techniques and principles of the non-linear analysis – electronics is based on the non-linear devices (semiconductor diodes, bipolar junction transistors, field-effect transistors and etc.). As semiconductor devices make the basis of the modern computers of all kind, it is important for engineers (majoring on computer science) to know something about non-linear analysis.

Lecture 17.

Distributed parameters circuits

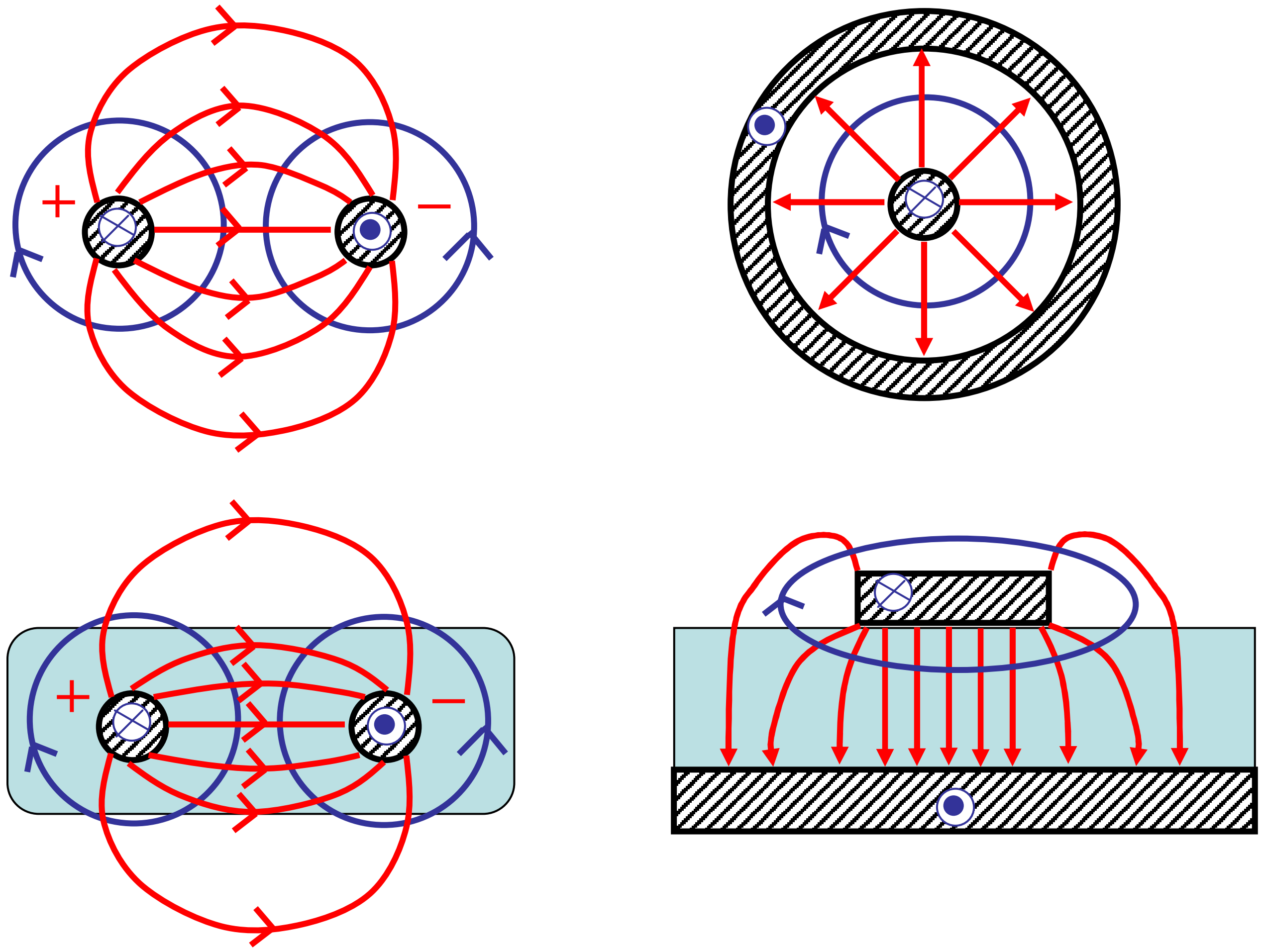
Introduction

Previous lectures describe lumped elements circuits. Lumped circuits consist of elements like capacitance, resistance, inductance and it doesn’t matter where they are placed, what are the sizes of such components, what are the distances between them. We just neglect the spatial distribution of the electrical circuit. It is reasonable because electrical signals, as well as electromagnetic field, travel through our circuits rather fast – with the speed of light (300 000 000 m/s). It takes electrical signal 3 ns to pass 1 m distance. As the least time constant of the circuit is greater than 30 ns (10 times greater!), it is possible to disregard the 3 ns delay. But, if the time constant of the circuit is comparable with the signal delay, it is necessary to take into account *distributed* character of the circuit.

Transmission lines

The most important case of the distributed circuits is a transmission line. Transmission line has long length and relatively small cross-section sizes. There are many different lines but there is one type that is in our scope – double (conductors) line. There are four major types of the double lines: open-wire line (1), coaxial cable (2), insulated-wire cable (3), microstrip line (4).

First type is used for power transmission and old-fashioned telegraph/telephone communications. Second one is widely used in computer networks, radio frequency devices. Third type is a very common type of the line for low power transmission and data transmission. Fourth is used in PCB (printed circuit boards) of different types.

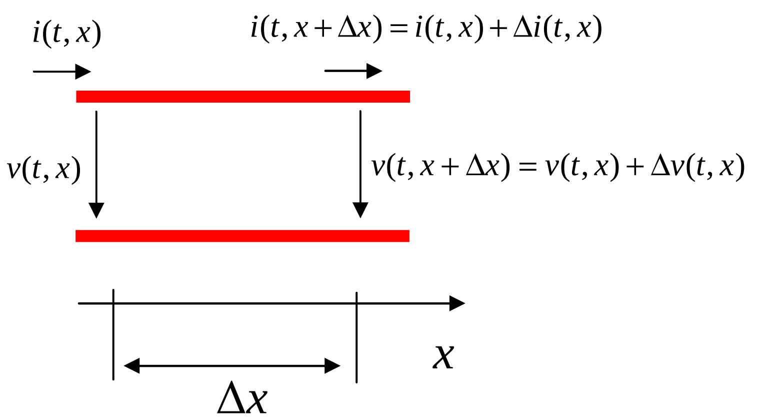


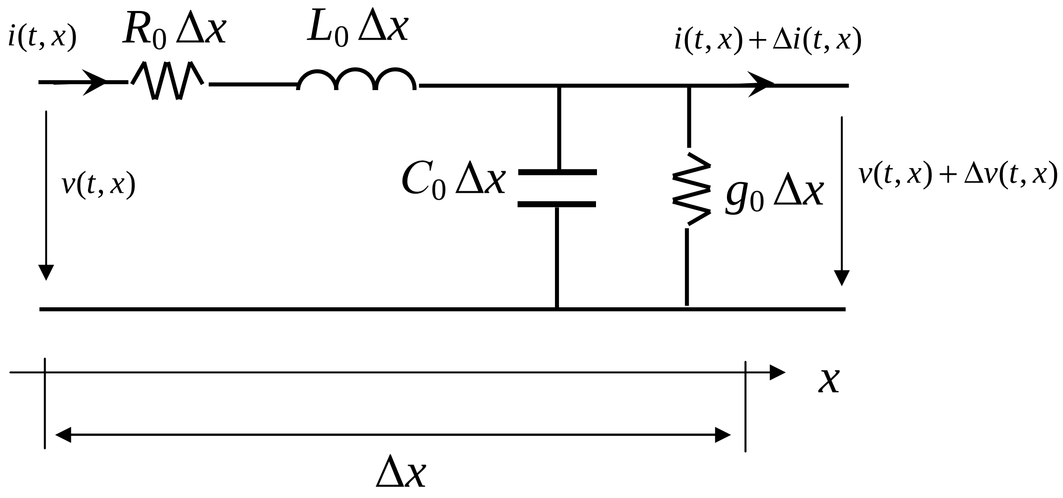
Note that all these lines have similar electromagnetic field distribution – electric field and magnetic field are lying on the same cross-section plane. This type of the electromagnetic wave is called TEM (transverse electric and magnetic fields) wave.

It is important that further analysis is valid for lines with the TEM waves. All other types of transmission lines (e.g. waveguides) are to be analyzed differently. In fact, our analysis is valid not only for double-lines, but for all TEM lines (with TEM wave).

Telegraph equations

Let's consider a lumped element model of the *infinitesimal* piece of the double transmission line:





This model consists of four elements – series connection of the longitudal resistance $R_0\Delta x$ and longitudal inductance $L_0\Delta x$; parallel connection of the transverse conductance $g_0\Delta x$ and transverse capacitance $C_0\Delta x$. Parameters of these elements R_0, L_0, C_0, g_0 are called **primary parameters** of the transmission line.

In details, primary parameters are: R_0 - **specific resistance** (Ω/m); L_0 - **specific inductance** (H/m); C_0 - **specific capacitance** (F/m); g_0 - **specific conductance** (S/m).

There are different methods of the primary parameters evaluation. It is important to note that specific resistance and specific inductance of the transmission line *depend on frequency* greatly.

As there is lumped model, it is possible to use KCL and KVL to obtain equations for the *voltage increment* Δv and the *current increment* Δi:

$$i(t,x) = i(t,x) + \Delta i(t,x) + C_0\Delta x \frac{d\left(v(t,x)+\Delta v(t,x)\right)}{dt} + g_0\Delta x\left(v(t,x)+\Delta v(t,x)\right)$$

$$v(t,x) = v(t,x) + \Delta v(t,x) + L_0\Delta x \frac{di(t,x)}{dt} + R_0\Delta x\ i(t,x)$$

One can obtain differential equations assuming $\Delta x \to dx$, $\Delta v \to dv$, $\Delta i \to di$:

$$\begin{cases} \dfrac{di(t,x)}{dx} = -C_0 \dfrac{dv(t,x)}{dt} - g_0\ u(t,x) \\ \dfrac{dv(t,x)}{dx} = -L_0 \dfrac{di(t,x)}{dt} - R_0\ i(t,x) \end{cases}$$

Note that *second order infinitesimals* $\Delta v \Delta i$ are neglected.

Then, substituting one equation into another one can obtain pair of differential equations:

$$\begin{cases} \dfrac{d^2 i}{dx^2} = C_0 L_0 \dfrac{d^2 i}{dt^2} + C_0 R_0 \dfrac{di}{dt} + g_0 L_0 \dfrac{di}{dt} + g_0 R_0 i \\ \dfrac{d^2 v}{dx^2} = C_0 L_0 \dfrac{d^2 v}{dt^2} + C_0 R_0 \dfrac{dv}{dt} + g_0 L_0 \dfrac{dv}{dt} + g_0 R_0 v \end{cases}$$

There are *linear partial differential equations (PDE) of the second order* – **telegraph equations**.

Secondary parameters of the transmission line

It is possible to use Laplace transform to obtain ODE with respect to *x* but in the *s*-domain:

$$\begin{cases} \dfrac{d^2 I(x,s)}{dx^2} = I(x,s)\left(s^2 C_0 L_0 + s\left(C_0 R_0 + g_0 L_0\right) + g_0 R_0 \right) \\ \dfrac{d^2 V(x,s)}{dx^2} = V(x,s)\left(s^2 C_0 L_0 + s\left(C_0 R_0 + g_0 L_0\right) + g_0 R_0 \right) \end{cases}$$

$$\begin{cases} \dfrac{d^2 I(x,s)}{dx^2} = I(x,s) Z_0(s) Y_0(s) \\ \dfrac{d^2 V(x,s)}{dx^2} = V(x,s) Z_0(s) Y_0(s) \end{cases}$$

$$Z_0(s) = sL_0 + R_0$$

$$Y_0(s) = sC_0 + g_0$$

Thus, there are homogeneous second order ODEs in spatial domain and they are written for the Laplace representations of the voltage and the current. Solution may be obtained as general solution with the roots of the characteristic equation:

$$\begin{cases} \dfrac{d^2 I(x,s)}{dx^2} = I(x,s)\gamma^2(s) \\ \dfrac{d^2 V(x,s)}{dx^2} = V(x,s)\gamma^2(s) \end{cases} \Rightarrow \lambda^2 = \gamma^2(s) \Rightarrow \begin{cases} \lambda_1 = -\gamma(s) \\ \lambda_2 = +\gamma(s) \end{cases}$$

$$\gamma^2(s) = Z_0(s) Y_0(s)$$

$$\begin{cases} V(x,s) = A_1(s) e^{-\gamma(s)x} + A_2(s) e^{+\gamma(s)x} \\ I(x,s) = B_1(s) e^{-\gamma(s)x} + B_2(s) e^{+\gamma(s)x} \end{cases}$$

Thus, $A_1(s)$, $A_2(s)$ and $B_1(s)$, $B_2(s)$ are the integration constants in spatial domain, but functions in the *s*-domain. These functions are to be obtained from **boundary conditions** (instead of *initial conditions*):

$$\begin{cases} V(x=0,s) = A_1(s) + A_2(s) \\ V(x=l,s) = A_1(s) e^{-\gamma(s)l} + A_2(s) e^{+\gamma(s)l} \\ I(x=0,s) = B_1(s) + B_2(s) \\ I(x=l,s) = B_1(s) e^{-\gamma(s)l} + B_2(s) e^{+\gamma(s)l} \end{cases}$$

The root of the characteristic equation $\gamma(s)$ is called **operational propagation coefficient**.

Current and voltage in the line are not independent functions. So, it is possible to find one pair of the integration constants through another pair:

$$Z_C(s) = \sqrt{\frac{Z_0(s)}{Y_0(s)}}$$

$$\begin{cases} V(x,s) = A_1(s) e^{-\gamma(s)x} + A_2(s) e^{+\gamma(s)x} \\ I(x,s) = \dfrac{A_1(s)}{Z_C(s)} e^{-\gamma(s)x} - \dfrac{A_2(s)}{Z_C(s)} e^{+\gamma(s)x} \end{cases}$$

Coefficient $Z_C(s)$ is called **operational characteristic impedance**.

Operational propagation coefficient and operational characteristic impedance are called **operational secondary parameters** of the transmission line.

It is very usual case when there is *active* circuit (real source, transmitter) connected at one end of the line ($x = 0$) and there is passive circuit (load, receiver) connected at another end of the line ($x = l$). By many reasons it is convenient to use another coordinate system: $y = l - x$. Transmitter is attached at $y = l$ and receiver is attached at $y = 0$. Operational solution of the telegraph equation may be rewritten as:

$$\begin{cases} V(y,s) = C_1(s)e^{+\gamma(s)y} + C_2(s)e^{-\gamma(s)y} \\ I(y,s) = \dfrac{C_1(s)}{Z_C(s)}e^{+\gamma(s)y} - \dfrac{C_2(s)}{Z_C(s)}e^{-\gamma(s)y} \end{cases}$$

And it is possible to use boundary condition at $y = 0$ (at the receiver end) to express one integration constant by another:

$$\begin{cases} V(y=0,s) = C_1(s) + C_2(s) \\ I(y=0,s) = \dfrac{C_1(s)}{Z_C(s)} - \dfrac{C_2(s)}{Z_C(s)} \end{cases}$$

$$\frac{V(y=0,s)}{I(y=0,s)} = Z_{load}(s) = Z_C(s)\frac{C_1(s)+C_2(s)}{C_1(s)-C_2(s)}$$

$$C_2(s) = N(s)C_1(s) \quad \rightarrow \quad N(s) = \frac{Z_{load}(s) - Z_C(s)}{Z_{load}(s) + Z_C(s)}$$

Coefficient $N(s)$ is called **operational reflection coefficient**. It depends on the operational load impedance and operational characteristic impedance.

Lossless line

Two of primary parameters (R_0 and g_0) are the models of the energy absorption. Every transmission line has certain rate of the *specific* energy absorption – there are always *ohmic losses* in the wires, for example. But the presence of these coefficients in the telegraph equations makes solution of the equations significantly difficult. For the most of transmission lines it is reasonable to start analysis (or synthesis) from the simplified model without specific power dissipation – **lossless line**.

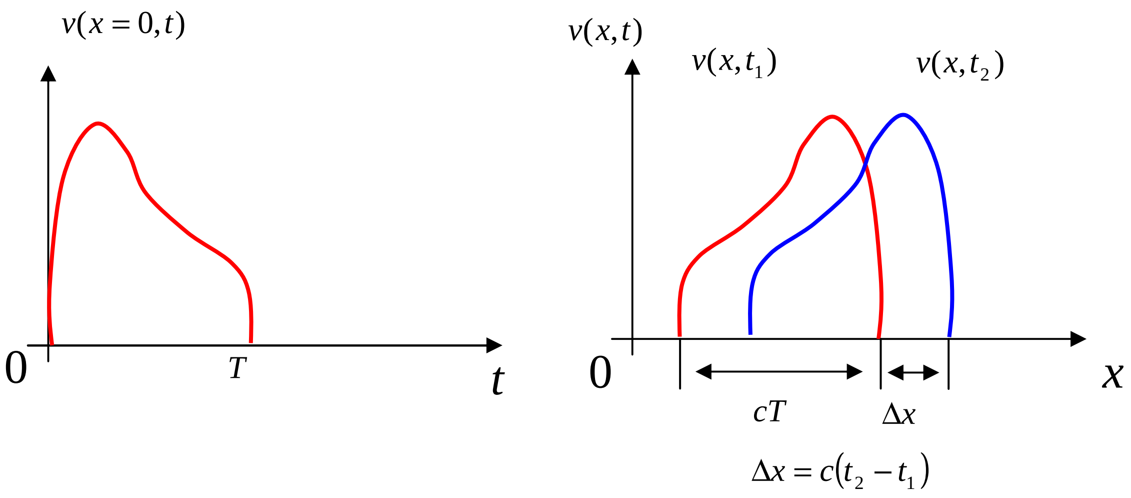
Lossless line has only pair of primary parameters – C_0 and L_0. Telegraph equations may be rewritten for the lossless line as:

$$\begin{cases} \dfrac{d^2 i}{dx^2} = C_0 L_0 \dfrac{d^2 i}{dt^2} \\ \dfrac{d^2 v}{dx^2} = C_0 L_0 \dfrac{d^2 v}{dt^2} \end{cases} \Rightarrow \begin{cases} \dfrac{d^2 i}{dx^2} = \dfrac{1}{c^2}\dfrac{d^2 i}{dt^2} \\ \dfrac{d^2 v}{dx^2} = \dfrac{1}{c^2}\dfrac{d^2 v}{dt^2} \end{cases}$$

This type of PDE is called *wave equation*. Solution of the wave equation is a function that is called *wave*. Wave is the function of the argument (t-x/c) where c is a *speed* of the wave. This value doesn't exceed speed of light c_0 and depends on primary parameters:

$$c = \frac{1}{\sqrt{L_0 C_0}}$$

Following figure illustrates wave propagation through the line:



AC analysis of the lossless line

As transmission line is described by the linear equations (ODE or PDE – doesn't matter), it is possible to use harmonic analysis (of the periodical signals with the Fourier series technique) and simple AC analysis (for sinusoidal signals). It isn't necessary to re-introduce solution of the telegraph equations that is obtained in operational form; it is necessary to substitute s by the $j\omega$:

$$\begin{cases} \dot{V}(y) = \dot{C}_1 e^{+\gamma y} + \dot{C}_2 e^{-\gamma y} \\ \dot{I}(y) = \dfrac{\dot{C}_1}{Z_C} e^{+\gamma y} - \dfrac{\dot{C}_2}{Z_C} e^{-\gamma y} \end{cases}$$

The first exponential function with the corresponded integration constant forms the *forward* or *falling* wave and the second exponent forms the *backward* or *reflected* wave.

Secondary parameters of the lossless line are:

$$\gamma = \sqrt{j\omega C_0 \, j\omega L_0} = j\omega\sqrt{C_0 L_0} = j\beta$$

$$Z_C = \sqrt{\frac{L_0}{C_0}}$$

Note that **propagation coefficient** γ has only imaginary part β that is called **phase coefficient**. **Characteristic impedance** is *active* and doesn't depend on frequency. So, solution of the telegraph equations may be rewritten as:

$$\begin{cases} \dot{V}(y) = \dot{C}_1 e^{+j\beta y} + \dot{C}_2 e^{-j\beta y} \\ \dot{I}(y) = \dfrac{\dot{C}_1}{Z_C} e^{+j\beta y} - \dfrac{\dot{C}_2}{Z_C} e^{-j\beta y} \end{cases}$$

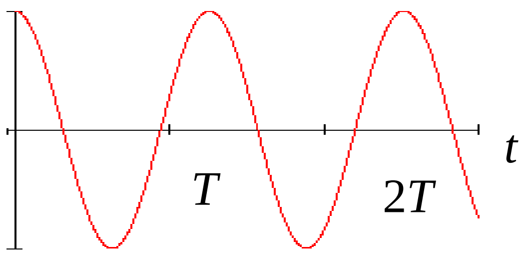
Note that characteristic impedance of the line doesn't fit the Ohm's law – it doesn't connect voltage and current, but forward wave of the voltage and forward wave of the current (or backward wave of the voltage and backward wave of the current). This is why one cannot use voltmeter and ammeter (or ohmmeter) to measure AC characteristic impedance directly.

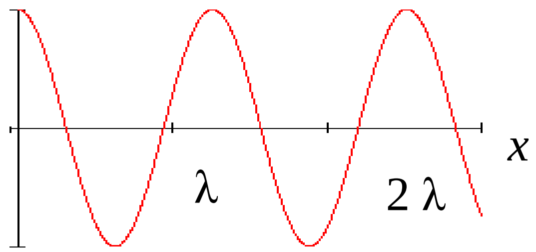
Reflection coefficient is a complex constant:

$$N = \frac{Z_{load} - Z_C}{Z_{load} + Z_C}$$

Note that magnitude of the reflection coefficient is *limited* by 1. Magnitude of the reflection coefficient is equal to 1 when there is *reactive*, *infinite* (open circuit) or *zero* (short circuit) load impedance. If load impedance contains finite real part (load contains active resistance), reflection coefficient magnitude is less than one.

If there is a sinusoidal signal in the line, *distance* between maximal points (or minimal points) of the signal is called **wavelength**:





Wavelength is connected with the period (or frequency) of the signal and **phase speed** c_P of the wave:

$$\lambda = c_P T = \frac{c_P}{f}$$

It is easy to find the connection between phase speed and phase coefficient:

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{c_P} = \frac{\omega}{c_P} \qquad c_P = \frac{1}{\sqrt{L_0 C_0}}$$

Note that phase speed of the wave in lossless line doesn't depend on frequency.

And it is possible to determine solution of the wave equation in other form:

$$\dot{V}(y) = \dot{V}_{load} b e^{j\beta y} + \dot{V}_{load} k \sin(\beta y + \psi) = \dot{V}_{run}(y) + \dot{V}_{stand}(y)$$

$$\downarrow$$

$$v(t, y) = V_{load\,a} b \sin(\omega t + \beta y) + V_{load\,a} k \sin(\omega t) \sin(\beta y + \psi)$$

It is a sum of the *standing wave* and *running wave*. And it may be shown that it is the running wave that carries power from transmitter to receiver.

As the solution of the wave equation may be represented as sum of the standing wave and the running wave, it is possible to connect forward wave and backward wave with them. **Standing wave ratio (SWR)** is introduced as:

$$SWR = \frac{V_{\max}}{V_{\min}} = \frac{I_{\max}}{I_{\min}} = \frac{1+|N|}{1-|N|}$$

If SWR = 1, there is running wave in the line. If SWR → ∞, there is standing wave in the line. Note that reactive, open or short load (load that doesn't absorb energy) corresponds to standing wave.

Input impedance of the transmission line may be calculated as a ratio of the input voltage $V(y = l)$ and input current $V(y = l)$:

$$Z_{IN}(y) = \frac{\dot{V}(y)}{\dot{I}(y)} = Z_C \frac{1+Ne^{-2\gamma y}}{1-Ne^{-2\gamma y}}$$

Input impedance depends on the reflection coefficient and length of the line. So it is possible to *adjust* input impedance of the line with the certain load by changing its length. *Matched* line has zero reflection coefficient (this is possible, if load impedance is equal to the characteristic impedance), thus input impedance of the matched line is equal to characteristic impedance and doesn't depend on the distance. *Matching* is a problem of the SWR reducing (or reflection reducing) for better power transmission.

Lossy line AC analysis

If it is impossible to neglect R_0 and g_0 or more precise analysis required, it is necessary to work with the full, lossy model of the line. Let's rewrite telegraph equations AC solution:

$$\begin{cases} \dot{V}(y) = \dot{C}_1 e^{+\alpha y} e^{+j\beta y} + \dot{C}_2 e^{-\alpha y} e^{-j\beta y} \\ \dot{I}(y) = \frac{\dot{C}_1}{Z_C} e^{+\alpha y} e^{+j\beta y} - \frac{\dot{C}_2}{Z_C} e^{-\alpha y} e^{-j\beta y} \end{cases}$$

$$\gamma = \sqrt{-\omega^2 C_0 L_0 + j\omega(C_0 R_0 + g_0 L_0) + g_0 R_0} = \alpha + j\beta$$

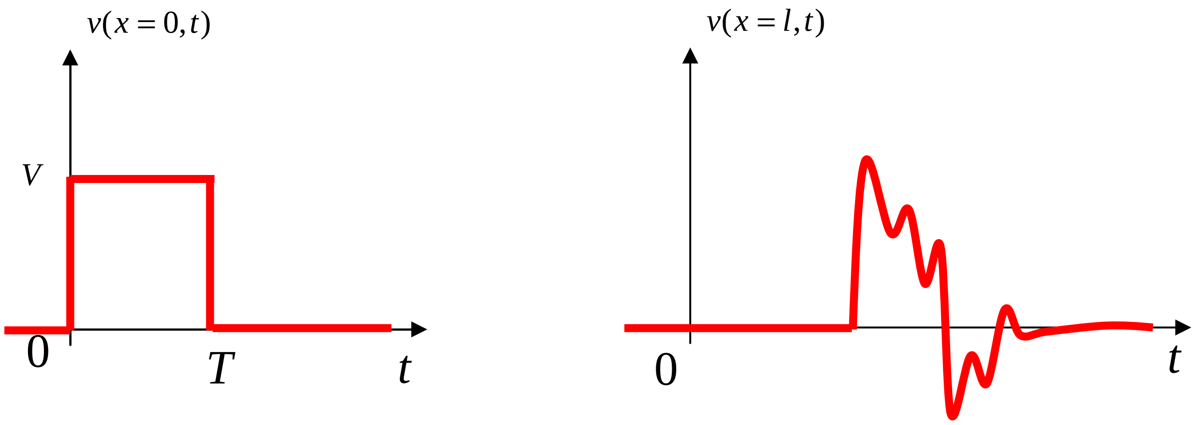
$$Z_C = \sqrt{\frac{j\omega L_0 + R_0}{j\omega C_0 + g_0}}$$

The difference between lossless line and lossy line is in the secondary parameters:

- propagation coefficient has non-zero real part α – **attenuation coefficient**;
- attenuation coefficient and phase coefficient depend on frequency (this means that phase speed depends on frequency);
- characteristic impedance is complex and depends on frequency.

Note that *exp*(-α*x*) declines and forward wave magnitude drops. And *exp*(-α*y*) declines so that, backward wave magnitude drops too. For example, if there is no reflected (backward) wave, voltage at the receiver end of the line in *exp*(-α*l*) times less than input voltage.

Another important property of the lossy line is a *dispersion* of the signal. Harmonic analysis (Fourier transform) leads to the frequency-domain representation of the input signal. As phase speed depends on frequency, different harmonics will have different delay time in the line. This affects the waveform of the input signal. Example of the signal dispersion is shown on the following figure:



It is possible to eliminate dispersion in a lossy line adjusting primary parameters – if there is certain condition satisfied:

$$\frac{R_0}{L_0} = \frac{g_0}{C_0} \leftrightarrow \begin{cases} \alpha(\omega) = const \\ \beta(\omega) = \omega \cdot const \leftrightarrow c_P = const \\ |Z_C(\omega)| = const \end{cases}$$

the phase speed doesn't depend on frequency. So, it is necessary to increase attenuation and power losses (gaining g_0) to save the waveform.

Conclusion

There was just a brief overview of the transmission line analysis and synthesis problem. Matching problem, time-domain analysis, mutual influences in a multi-wire cable – all these problems are out of our scope. Important thing is that 50 Ω cable is a cable with 50 Ω characteristic impedance (and it is assumed that cable is lossless lone). If the cable is marked as 50 Ω, 6 dB, it means that there is constant (in a working frequency range) characteristic impedance and attenuation is equal to 6 dB (or 2 times).

Literature

Clayton R. Paul Analysis of linear circuits -NY.: McGraw-Hill, 1989, 794 p.